1 $1^2 + 2^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1)$

BS. $n = 1$

$LS = 1$

$RS = \frac{1}{6} \cdot 1 \cdot (1+1)(2 \cdot 1 + 1) = 1$

IH. $1^2 + 2^2 + \dots + k^2 = \frac{1}{6} k(k+1)(2k+1)$

IS. $1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$

$\Leftrightarrow \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$

$= (k+1)\left(\frac{1}{6} k(2k+1) + (k+1)\right)$

$= (k+1)\left(\frac{1}{3} k^2 + \frac{1}{6} k + k + 1\right)$

$= (k+1) \frac{1}{6} (2k^2 + 7k + 6)$

$= (k+1) \frac{1}{6} (2k(k+2) + 3(k+2))$

$= \frac{1}{6} (k+1)(k+2)(2k+3)$

$LS = RS \Rightarrow$ True

2. a) $n^2 < 2^n, \forall n \geq 5$.

BS: $n = 5$

$LS = 25$

$RS = 32$

IH: $k^2 < 2^k \quad \forall k \geq 5$

IS: $(k+1)^2 < 2^{k+1}$

$\Leftrightarrow k^2 + 2k + 1 < 2 \cdot 2^k$

$k^2 + 2k + 1 < k^2 + 2k + k = k^2 + 3k < k^2 + k^2 = 2k^2$

$2k^2 < 2 \cdot 2^k = 2^{k+1}$

$\Rightarrow$ True

2b) $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}, \forall n \geq 2$

BS: $n = 2$

$LS = 1 + \frac{\sqrt{2}}{2}$

$RS = \sqrt{2}$

IH: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} > \sqrt{k}, \forall k \geq 2$

IS $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

$\Leftrightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$

$\sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}} > \frac{\sqrt{k}\sqrt{k+1}}{\sqrt{k+1}} = \sqrt{k+1}$

$\Rightarrow$ True

3. a) $P_1 = \frac{P_0^2}{2}$

$P_2 = \frac{P_1^2}{2} = \frac{P_0^4}{8} =$ $\qquad P_{n+1} = \frac{P_n^2}{2}$

$P_3 = \frac{P_2^2}{2} = \frac{P_0^8}{128}$

$P_4 = \frac{P_3^2}{2} = \frac{P_0^{16}}{32768}$

$P_5 = \frac{P_4^2}{2} = \frac{P_0^{32}}{2147483648}$

b) Let $P_0 = x$

$P_1 = \frac{x^2}{2}$ $\quad P_2 = \frac{x^4}{8}$ $\quad P_3 = \frac{x^{16}}{32768}$

$\Rightarrow P_n = \left(\frac{1}{2}\right)^{2^n - 1} \cdot x^{2^n} = 2^{1-2^n} \cdot x^{2^n}$

c) BS: $n = 0$

RS $2^{1-2^0} \cdot x^{2^0} = 1$

LS: 0

IH: $2^{1-2^k} \cdot x^{2^k} = P_k$

IS. $P_{k+1} = 2^{1-2^{(k+1)}} \cdot x^{2^{k+1}}$

$= 2^{1-2^k \cdot 2} \cdot x^{2^k 2}$

$= \cancel{2^{1-2k} \cdot 2^{-2k} \cdot x^{2k} \cdot x^{2k}}$

$= 2^{1-2^k} \cdot 2^{-2^k} \cdot x^{2^k} \cdot x^{2^k}$

$= P_k \cdot \dfrac{x^{2^k}}{2^{2^k}}$

4. a) $y_{n+2} - 3y_{n+1} - 10y_n = 0 \quad (y_n = r^n)$

$\Leftrightarrow r^{n+2} - 3r^{n+1} - 10r^n = 0$

$\Leftrightarrow r^n(r^2 - 3r - 10) = 0$

$r^2 - 3r - 10 = 0$

$\Leftrightarrow \left[\begin{array}{l} r_1 = -2 \\ r_2 = 5 \end{array}\right.$

$y_n = c_1(-2)^n + c_2 5^n$

4b) $9y_{n+2} - 6y_{n+1} + y_n = 0 \ , y_0 = 2 , y_1 = 3$

$\Leftrightarrow r^n(9r^2 - 6r + 1) = 0$

$\Rightarrow \quad 9r^2 - 6r + 1 = 0$

$\Leftrightarrow r = \frac{1}{3}$

$y_n = c_1\left(\frac{1}{3}\right)^n + c_2 \cdot n \left(\frac{1}{3}\right)^n$

$y_0 = c_1 \pm 2$

$y_1 = c_1 \cdot \frac{1}{3} + c_2 \cdot \frac{1}{3} = 3 \Leftrightarrow c_2 = 7$

$y_n = 2\left(\frac{1}{3}\right)^n + 7 \cdot n \left(\frac{1}{3}\right)^n$

5a) $y_{n+2} - 3y_{n+1} + 2y_n = 3^n$

$r^2 - 3r + 2 = 0$

$\Leftrightarrow \left[\begin{array}{l} r_1 = 1 \\ r_2 = 2 \end{array}\right.$

$y_{n,h} = c_1 + c_2 2^n$

$y_{n,p} = A3^n$

$A3^{n+2} - 3A3^{n+1} + 2A3^n = 3^n$

$9A3^n - 9A3^n + 2A3^n = 3^n$

$\Leftrightarrow 3^n(9A - 9A + 2A) = \cancel{0}\,1$

$\Leftrightarrow \quad A = \cancel{0}\ \frac{1}{2}$

$\Rightarrow y_{n,p} = \cancel{0}\ \frac{1}{2}3^n$

$\Rightarrow y_n = c_1 + c_2 2^n + \frac{1}{2}3^n$

5b) $y_{n+2} - 3y_{n+1} + 2y_n = 5$

$r^2 - 3r + 2r = 0$

$\Leftrightarrow \begin{cases} r_1 = 1 \\ r_2 = 2 \end{cases}$

$y_{n,h} = c_1 + c_2 2^n \qquad y_{n,p} = An$

$A(n+2) - 3(A(n+1)) + 2An = 5$

$\Leftrightarrow An + 2A - 3An - 3A + 2An = 5$

$\Leftrightarrow -A = 5$

$\Leftrightarrow A = -5$

$y_n = c_1 + c_2 2^n - 5n$

6) $y_{n+3} - 6y_{n+2} + 5y_{n+1} + 12y_n = 6n$

$r^3 - 6r^2 + 5r + 12 = 0$

$\Leftrightarrow \begin{cases} x_1 = -1 \\ x_2 = 3 \\ x_3 = 4 \end{cases}$

$y_{n,h} = c_1(-1)^n + c_2 3^n + c_3 4^n$

$y_{n,p} = An + B$

$(A(n+3) + B) - 6(A(n+2) + B) + 5(A(n+1) + B) + 12(An + B) = 6n$

$An + 3A + B - 6An - 12A - 6B + 5An + 5A + 5B + 12An + 12B = 6n$

$12An - 4A + 12B = 6n$

$\Leftrightarrow \begin{cases} A = 1/2 \\ B = 1/6 \end{cases} \qquad \Leftrightarrow y_n = c_1(-1)^n + c_2 3^n + c_3 4^n + \frac{1}{2}n + \frac{1}{6}$