1.

$$\begin{aligned}\binom{n-1}{k}+\binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!}+\frac{(n-1)!}{(k-1)!(n-k)!}\\ &= (n-1)!\left(\frac{1}{k!(n-k-1)!}+\frac{1}{(k-1)!(n-k)!}\right)\\ &= (n-1)!\left(\frac{n-k}{k!(n-k)!}+\frac{k}{k!(n-k)!}\right)\\ &= (n-1)!\frac{n}{k!(n-k)!}\\ &= \frac{n!}{k!(n-k)!}\\ &= \binom{n}{k}.\end{aligned}$$

2. $(\forall n \geq 1)\ 1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$

Base $n=1$: Left side $1^2=1$, right side $\frac{1}{6}\cdot 1\cdot 2\cdot 3$

Induction step: Suppose that the claim holds for $n=k$, that is,

$$1^2+2^2+3^2+\cdots+k^2 = \frac{1}{6}k(k+1)(2k+1)$$

Let $n=k+1$. Now

$$\begin{aligned}1^2+2^2+\cdots+k^2+(k+1)^2 &= \frac{1}{6}k(k+1)(2k+1)+\frac{6}{6}(k+1)^2\\ &= \frac{1}{6}(k+1)[k(2k+1)+6(k+1)]\\ &= \frac{1}{6}(k+1)[2k^2+k+6k+6)]\\ &= \frac{1}{6}(k+1)(2k^2+7k+6)\\ &= \frac{1}{6}(k+1)(k+2)(2k+3)\\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)\end{aligned}$$

that is, the claim holds for $n=k+1$.

3. $(\forall n \geq 2)\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)\cdot n} = \frac{n-1}{n}$

Base $n=2$: Left side $\frac{1}{1\cdot 2}=\frac{1}{2}$, right side $\frac{1}{2}$.

Induction step: Suppose that the claim holds for $n=k$, that is,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k-1)\cdot k} = \frac{k-1}{k}$$

Let $n = k + 1$. Now

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1) \cdot k} + \frac{1}{k \cdot (k+1)} =$$

$$\frac{k-1}{k} + \frac{1}{k \cdot (k+1)} = \frac{(k-1)(k+1)+1}{k(k+1)} = \frac{k^2}{k(k+1)} = \frac{k}{k+1}$$

Thus, the claim holds also for $n = k + 1$.

4. $(\forall n \geq 2)\, \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1$

Let $n \geq 2$. Because $n - 1 < n$, we have $(n-1)n < n \cdot n = n^2$ and

$$\frac{1}{n^2} < \frac{1}{n(n-1)}$$

We obtain

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} = \frac{n-1}{n}$$

The last equality follows from Exercise 3. Because $n - 1 < n$, dividing by n gives

$$\frac{n-1}{n} < 1,$$

which completes the proof.

5. $(\forall \geq 1)\, (1+x)^n \geq 1 + nx$, where $x \geq -1$
Note first that $x \geq -1$ is essential. If $x = -50$ and $n = 3$, then $(1-50)^3 = -117649$ and $1 - 3 \cdot 50 = -149$, and the claim does not hold. Because $1 + x \geq 0$, we can multiply numbers by it and the order of $\geq$-sign does not change in (**).

Base $n = 1$: Left side $(1+x)^1 = 1 + x$, right side $1 + 1 \cdot x = 1 + x$.

Induction step: Suppose that the claim holds for $n = k$, that is,

$$(1+x)^k \geq 1 + kx$$

Let $n = k + 1$.

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x) \overset{(**)}{\geq} (1+kx)(1+x) = 1 + x + kx + kx^2 \\ &= 1 + x(k+1) + kx^2 \geq 1 + (k+1)x,\end{aligned}$$

since $kx^2 \geq 0$. We have shown that the claim is true also for $k = n + 1$.

6. $(\forall n \geq 0)\, F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

Base $n = 0$: Left side $F_0 = 0$, right side $F_2 - 1 = 1 - 1 = 0$.

Induction step: Suppose that the claim holds for $n = k$, that is,

$$F_0 + F_1 + F_2 + \cdots + F_k = F_{k+2} - 1.$$

Let $n = k + 1$. Now

$$\begin{aligned}F_0 + F_1 + F_2 + \cdots + F_k + F_{k+1} &= (F_{k+2} - 1) + F_{k+1} - 1 \\ &= F_{k+1} + F_{k+2} = F_{k+3} - 1 = F_{(k+1)+2} - 1.\end{aligned}$$