Mathematics A
Assignment two / Solutions

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1. Let U be a set and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be nonempty families of subsets of U such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Show (=prove) that the following inclusions hold:

(a) $\bigcup \mathcal{F}_1 \subseteq \bigcup \mathcal{F}_2$
(b) $\bigcap \mathcal{F}_2 \subseteq \bigcap \mathcal{F}_1$

Solution. (a) Let $x \in \bigcup \mathcal{F}_1$. This means that there is $A \in \mathcal{F}_1$ such that $x \in A$. Because $\mathcal{F}_1 \subseteq \mathcal{F}_2$, there exists $A \in \mathcal{F}_2$ such that $x \in A$. Therefore, $x \in \bigcup \mathcal{F}_2$ and the claim is proved.

(b) We prove $\left(\bigcap \mathcal{F}_1\right)^c \subseteq \left(\bigcap \mathcal{F}_2\right)^c$. This is equivalent to the original claim, but is easier to prove. As it is written is the lecture notes: "$A \subseteq B$ can be sometimes shown easier by showing that $B^c \subseteq A^c$". Assume that $x \in \left(\bigcap \mathcal{F}_1\right)^c$. This means that $x \notin \bigcap \mathcal{F}_1$. So, there is $A \in \mathcal{F}_1$ such that $x \notin A$. Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, there is $A \in \mathcal{F}_2$ such that $x \notin A$. Therefore, $x \notin \bigcap \mathcal{F}_1$ and $x \in \left(\bigcap \mathcal{F}_2\right)^c$. The claim is proved.

2. Let U be a set and let $\emptyset \neq \mathcal{F} \subseteq \wp(U)$ be a nonempty family of subsets of U. Prove the following equalities:

(a) $(\bigcap \mathcal{F})^c = \bigcup \{A^c \mid A \in \mathcal{F}\}$
(b) $(\bigcup \mathcal{F})^c = \bigcap \{A^c \mid A \in \mathcal{F}\}$

Recall that the complement of any $X \subseteq U$ is defined by $X^c = U \setminus X$.

Solution. Let $x \in U$.

$$x \in \left(\bigcap \mathcal{F}\right)^c \iff x \notin \bigcap \mathcal{F} \iff (\exists A \in \mathcal{F})\, x \notin A \iff (\exists A \in \mathcal{F}) x \in A^c \iff x \in \bigcup \{A^c \mid A \in \mathcal{F}\}.$$

This proves (a). Case (b) is rather similar:

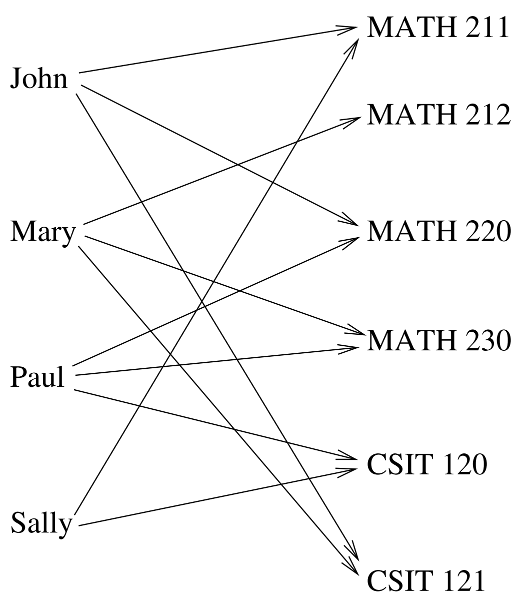
$$x \in \left(\bigcup \mathcal{F}\right)^c \iff x \notin \bigcup \mathcal{F} \iff (\forall A \in \mathcal{F})\, x \notin A \iff (\forall A \in \mathcal{F}) x \in A^c \iff x \in \bigcap \{A^c \mid A \in \mathcal{F}\}.$$

3. The courses taken by John, Mary, Paul, and Sally are listed below:

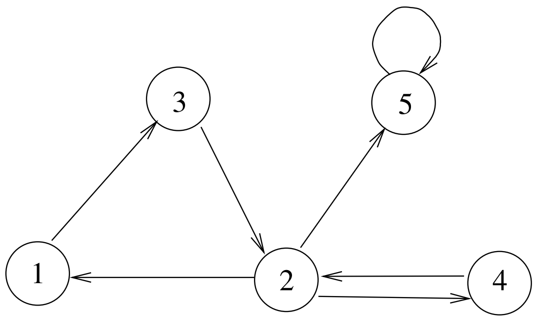
John: MATH 211, CSIT 121, MATH 220
Mary: MATH 230, CSIT 121, MATH 212
Paul: CSIT 120, MATH 230, MATH 220
Sally: MATH 211, CSIT 120

Give a graphical representation of the relation R defined as $a\, R\, b$ if student a is taking course b.

Solution.



4. Write the set of ordered pairs for the relation represented by the following directed graph:



Solution.

$$\{(1,3),(2,1),(2,4),(2,5),(3,2),(4,2),(5,5)\}$$

5. Let R be a binary relation on the set $\wp(\{a,b\})$ defined so that $(A,B) \in R$ holds if $A \cap B = \emptyset$. Write out the relation R.

Solution.

$$R = \{(\emptyset,\emptyset),(\emptyset,\{a\}),(\emptyset,\{b\}),(\emptyset,\{a,b\}),(\{a\},\emptyset),(\{b\},\emptyset),(\{a,b\},\emptyset),(\{a\},\{b\}),(\{b\},\{a\})\}$$

6. Let A, B, C be sets. Prove the following equalities:

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Solution. We show that the elements in left-hand side are the same as the elements in the right-hand side.

$$\begin{gathered}(a,b) \in A \times (B \cap C) \iff a \in A \text{ and } b \in (B \cap C) \iff a \in A \text{ and } (b \in B \text{ and } b \in C) \\ \iff (a \in A \text{ and } b \in B) \text{ and } (a \in A \text{ and } b \in C) \iff (a,b) \in A \times B \text{ and } (a,b) \in A \times C \\ \iff (a,b) \in (A \times B) \cap (A \times C).\end{gathered}$$

This proves (a). The proof for (b) is similar:

$$\begin{gathered}(a,b) \in A \times (B \cup C) \iff a \in A \text{ and } b \in (B \cup C) \iff a \in A \text{ and } (b \in B \text{ or } b \in C) \\ \iff (a \in A \text{ and } b \in B) \text{ or } (a \in A \text{ and } b \in C) \iff (a,b) \in A \times B \text{ or } (a,b) \in A \times C \\ \iff (a,b) \in (A \times B) \cup (A \times C).\end{gathered}$$