Matrices

A **matrix** is a rectangular array of numbers for which operations such as addition and multiplication are defined. In this course, we mainly consider matrices whose elements are real numbers. They are called **real matrix**.

$$A = \begin{bmatrix} -1.3 & 0.6 \\ 20.4 & 5.5 \\ 9.7 & -6.2 \end{bmatrix}.$$

The numbers in the matrix are called **entries** or **elements**. The horizontal and vertical lines of entries in a matrix are called **rows** and **columns**, respectively.

The **size** of a matrix is defined by the number of rows and columns that it contains. A matrix with m rows and n columns is called an $m \times n$ matrix, or m-by-n matrix, while m and n are called its **dimensions**. For example, the matrix A above is a 3×2 matrix. Note that *rows* are mentioned first and *columns* are mentioned second.

Matrices with a single row are called **row matrices**. Those with a single column are called **column matrices**. A matrix with the same number of rows and columns is called a **square matrix**.

Matrices are commonly written in square brackets or parentheses:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

These matrices are also denoted by

$$[a_{ij}]_{m\times n} \quad \text{and} \, (a_{ij})_{m\times n}$$

In this course, we mostly use square brackets.

A Operations

It the matrices are of the same type, they can be summed together. The **sum** $A + B$ of two m-by-n matrices A and B is calculated entrywise:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

For example,

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{bmatrix}$$

We can expess a matrix sum also in more concise way:

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}].$$

This should be understood in such a way that the sum of the matrices $[a_{ij}]$ and $[b_{ij}]$ is a matrix whose entry in position (i, j) is $a_{ij} + b_{ij}$.

Proposition 1. *Let* $A = [a_{ij}]_{m\times n}$ *and* $B = [b_{ij}]_{m\times n}$.

(a) *Matrix addition is* commutative*:* $A + B = B + A$.

(b) *Matrix addition is* associative*:* $(A + B) + C = A + (B + C)$.

Proof. (a) The sum of real numbers is commutative. Therefore,

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}].$$

(b) The sum of real numbers is commutative. This means that:

$$\begin{aligned}([a_{ij}] + [b_{ij}]) + [c_{ij}] &= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}] = [a_{ij} + (b_{ij} + c_{ij})] \\ &= [a_{ij}] + [b_{ij} + c_{ij}] = [a_{ij}] + ([b_{ij}] + [c_{ij}]) .\end{aligned}$$ □

Associativity means that if we are adding several matrices, we can order them any way we wish:

$$\left(\begin{bmatrix} 1 & -2 \\ 5 & -11 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 9 & 8 \end{bmatrix}\right) + \begin{bmatrix} 4 & 3 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 14 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 7 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix}1 & -2\\5 & -11\end{bmatrix}+\left(\begin{bmatrix}0 & 2\\9 & 8\end{bmatrix}+\begin{bmatrix}4 & 3\\-7 & 6\end{bmatrix}\right)=\begin{bmatrix}1 & -2\\5 & -11\end{bmatrix}+\begin{bmatrix}4 & 5\\2 & 14\end{bmatrix}=\begin{bmatrix}5 & 3\\7 & 3\end{bmatrix}.$$

The **subtraction of matrices** is possible if the number of rows and columns of both the matrices are the same. While subtracting two matrices, we subtract the elements in each row and column from the corresponding elements in the row and column of the other matrix. We can give this as a formula

$$A-B=[a_{ij}]-[b_{ij}]=[a_{ij}-b_{ij}]\,.$$

Of course, matrix subtraction is **not** commutative, that is, $A-B=B-A$ does not generally hold.

Example 2.

$$\begin{bmatrix}1 & 3\\1 & 0\\1 & 2\end{bmatrix}-\begin{bmatrix}0 & 0\\7 & 5\\2 & 1\end{bmatrix}=\begin{bmatrix}1-0 & 3-0\\1-7 & 0-5\\1-2 & 2-1\end{bmatrix}=\begin{bmatrix}1 & 3\\-6 & -5\\-1 & 1\end{bmatrix}$$

On the other hand,

$$\begin{bmatrix}0 & 0\\7 & 5\\2 & 1\end{bmatrix}-\begin{bmatrix}1 & 3\\1 & 0\\1 & 2\end{bmatrix}\begin{bmatrix}-1 & -3\\6 & 5\\1 & -1\end{bmatrix}$$

A **zero matrix 0** is a matrix all of whose entries are 0. For instance, the following are zero matrices:

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

There is an $m\times n$ zero matrix for every pair of positive dimensions m and n. If we add the $m\times n$ zero matrix to another $m\times n$ matrix A, we get A.

If is clear that for any matrix, the result of the subtraction $A-A$ is a zero matrix. Similarly, if $\mathbf{0}$ is a zero matrix of the same size than A, then $A=A+\mathbf{0}$ and $\mathbf{0}+A$.

The **scalar multiplication** of a matrix A with a number c gives another matrix of the same size as A. It is denoted by cA, whose entries are defined by:

$$cA=c\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1m}\\a_{21} & a_{22} & \cdots & a_{2m}\\\vdots & \vdots & \ddots & \vdots\\a_{n1} & a_{n2} & \cdots & a_{nm}\end{bmatrix}=\begin{bmatrix}ca_{11} & ca_{12} & \cdots & ca_{1m}\\ca_{21} & ca_{22} & \cdots & ca_{2m}\\\vdots & \vdots & \ddots & \vdots\\ca_{n1} & ca_{n2} & \cdots & ca_{nm}\end{bmatrix}$$

In a concise way, we can write:

$$c[a_{ij}]=[c\,a_{ij}].$$

Example 3.

$$2\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix} = \begin{bmatrix}2\cdot 2 & 2\cdot -1\\ 2\cdot 0 & 2\cdot 3\end{bmatrix} = \begin{bmatrix}4 & -1\\ 0 & 6\end{bmatrix}$$

Note also that for matrices A and B of the same size:

$$A - B = A + (-1)B.$$

The **transpose** of a matrix A, denoted by A^{T} be constructed by:

- write the rows of A as the columns of A^{T}, or equivalently,
- write the columns of A as the rows of A^{T}

This can be expressed in form

$$[a_{ij}]^{\mathrm{T}} = [a_{ji}].$$

This means that the transposed matrix of $[a_{ij}]$ is a matrix such that in position a_{ij} is the element a_{ji}. Note also that if A is an $m \times n$ matrix, then A^{T} is an $n \times m$ matrix.

Example 4.

$$\begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}1\\ 2\end{bmatrix}, \qquad \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}, \qquad \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$$

Proposition 5. *Let A and B be matrices and $c \in \mathbb{R}$.*

(a) $\left(A^{\mathrm{T}}\right)^{\mathrm{T}} = A$

(b) $(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$

(c) $(cA)^{\mathrm{T}} = c\left(A^{\mathrm{T}}\right)$.

Proof. (a) Because transpose changes row to columns and columns to rows, then the second transpose reverses the action of the first one, that is, $\left([a_{ij}]^{\mathrm{T}}\right)^{\mathrm{T}} = [a_{ji}]^{\mathrm{T}} = [a_{ij}]$.
(b) $[a_{ij} + b_{ij}]^{\mathrm{T}} = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = [a_{ij}]^{\mathrm{T}} + [b_{ji}]^{\mathrm{T}}$.
(c) $[ca_{ij}]^{\mathrm{T}} = [ca_{ji}] = c[a_{ji}] = c[a_{ij}]^{\mathrm{T}}$. □

Let A be a $m \times n$-matrix and B be a $n \times p$-matrix, that is,

$$A = \begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1p}\\ b_{21} & b_{22} & \cdots & b_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{np}\end{bmatrix}$$

The **product** AB is the $m \times p$ matrix:

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp}, \end{bmatrix}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

This means that when computing the entry (i, j) of the product, we take the for i of the matrix A and the column of the matrix j and multiply their 'corresponding' entries. Then we take the sum of the products.

$$C = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1p} + \cdots + a_{2n}b_{np} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

The product AB is defined if and only if the number of columns in A equals the number of rows in B!

For example,

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 28 \\ 9 & 28 \end{bmatrix}$$

The element 28 in the lower right corner is obtained by

$$1 \cdot 2 + 5 \cdot 4 + 2 \cdot 3 = 2 + 20 + 6$$

If we multiply a matrix by a zero matrix, we get a zero matrix. Matrix multiplication is *associative*: If A, B, and C are matrices and their dimensions are compatible for multiplication, then

$$(AB)C = A(BC).$$

This is not proved in lectures; see this for details.

Example 6. The matrix multiplication is **not** commutative in general. That is, $AB \neq BA$ does not generally hold. The product AB may be defined without BA being defined, namely if A and B are m-by-n and n-by-k matrices, and $m \neq k$. Note also that even the products are AB and BA are defined, they generally need not be equal:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix},$$

whereas

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$$

If the product exists, the transpose of a product of matrices is the product, in the reverse order, of the transposes of the factors:

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$$

The proof of this formula is left as an exercise.

B Inverse of a matrix

The **identity matrix** of size n is the $n \times n$ square matrix I_n with ones on the main diagonal and zeros elsewhere:

$$I_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \cdots$$

If I_n is the $n \times n$ identity matrix for some n, and A is a matrix which is compatible for multiplication with A, then

$$AI_n = A \quad \text{and} \quad I_n A = A.$$

The **inverse** of an $n \times n$ matrix A is a matrix A^{-1} which satisfies

$$AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n.$$

There is no such thing as matrix division in general, but if A has an inverse, we can simulate division by multiplying by A^{-1} . This is often useful in solving matrix equations. *Note that we often denote I_n simply by I if there is no danger of confusion.*

Proposition 7. *Let A and B be invertible $n \times n$ matrices.*

(a) $(AB)^{-1} = B^{-1}A^{-1}$.

(b) $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$.

Proof. (a) Now

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

and

$$(AB)(B^{-1}A^{-1}) = A^{-1}(B^{-1}B)A = AIA^{-1} = AA^{-1} = I.$$

(b) Similarly,

$$A^{\mathrm{T}}(A^{-1})^{\mathrm{T}} = \big(A^{-1}A\big)^{\mathrm{T}} = I^{\mathrm{T}} = I \quad \text{and} \quad (A^{-1})^{\mathrm{T}}A^{\mathrm{T}} = \big(AA^{-1}\big)^{\mathrm{T}} = I^{\mathrm{T}} = I. \qquad \square$$

By using the following lemma, we can find easily the inverse of a 2×2 matrix.

Lemma 8. *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $ad - bc \neq 0$, then the inverse of A is

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. Let us denote

$$B = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

To show that B is the inverse of A, we have to check that $AB = I$ and $BA = I$:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad-bc)} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly,

$$BA = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{(ad-bc)} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

□

Because not every matrix has an inverse, it is important to know the answer to the following questions:

- When a matrix has an inverse?
- How to find the inverse, if there is one?

We will try to answer to these two questions next.

C Elementary matrices and row operations

In this short section, we consider so-called elementary row operations. Elementary row operations are used in Gauss–Jordan elimination method to reduce a matrix to the reduced row echelon form. We will consider *reduced row echelon form* and *Gauss–Jordan elimination* in detail in the following section. Later we will see how using Gauss–Jordan elimination we can solve systems of equations and find the inverse of a matrix (if that exists).

There are three types of **elementary row operations**:

- Multiplying a row by a nonzero number
- Swapping two rows
- Adding a multiple of one row to another row

An **elementary matrix** is a matrix which differs from the identity matrix by one single elementary row operation. Multiplying a matrix A by an elementary matrix E (on the **left side**) causes A to undergo the elementary row operation described below:

Row multiplication Multiplies all elements on row i by m where m is a non-zero real number. The corresponding elementary matrix is a diagonal matrix, with diagonal entries 1 everywhere except in the ith position, where it is m.

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & m & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

Multiplies all elements on row i by m

Row switching Switches all elements on row i with their counterparts on row j. The corresponding elementary matrix is obtained by swapping row i and row j of the identity matrix.

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

Row addition Adds to row i the row j multiplied by a scalar m. The corresponding elementary matrix is the identity matrix but with an m in the (i, j) position:

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & m & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

If E is an elementary matrix, as described below, to apply the elementary row operation to a matrix A, one multiplies A by the elementary matrix on the left, EA.

Example 9. Each elementary matrix is invertible and the inverse matrix performs the "inverse operation":

Row multiplication Multiplying on the left by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

multiplies row 2 by 17. Its inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{17} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

divides row 2 by 17. For instance,

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} & 1 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} & 1 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} \\ 0 \cdot a_{11} + 17 \cdot a_{21} + 0 \cdot a_{31} & 0 \cdot a_{12} + 17 \cdot a_{22} + 0 \cdot a_{32} & 0 \cdot a_{13} + 17 \cdot a_{23} + 0 \cdot a_{33} \\ 0 \cdot a_{11} + 0 \cdot a_{21} + 1 \cdot a_{31} & 0 \cdot a_{12} + 0 \cdot a_{22} + 1 \cdot a_{32} & 0 \cdot a_{13} + 0 \cdot a_{23} + 1 \cdot a_{33} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 17a_{21} & 17a_{22} & 17a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{aligned}$$

Row switching Multiplying on the left by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

swaps row 2 and row 3. The inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

swaps row 2 and row 3. For example,

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} & 1 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} & 1 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} \\ 0 \cdot a_{11} + 0 \cdot a_{21} + 1 \cdot a_{31} & 0 \cdot a_{12} + 0 \cdot a_{22} + 1 \cdot a_{32} & 0 \cdot a_{13} + 0 \cdot a_{23} + 1 \cdot a_{33} \\ 0 \cdot a_{11} + 1 \cdot a_{21} + 0 \cdot a_{31} & 0 \cdot a_{12} + 1 \cdot a_{22} + 0 \cdot a_{32} & 0 \cdot a_{13} + 1 \cdot a_{23} + 0 \cdot a_{33} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & 17a_{23} \end{bmatrix} \end{aligned}$$

Row addition Multiplying on the left by

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

adds 2 times row 3 to row 1. Note that the element $a_{13} = 2$. Its inverse

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

subtracts 2 times row 3 from row 1. For instance,

$$
\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}
\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}
$$

$$
= \begin{bmatrix}1\cdot a_{11}+0\cdot a_{21}+2\cdot a_{31} & 1\cdot a_{12}+0\cdot a_{22}+2\cdot a_{32} & 1\cdot a_{13}+0\cdot a_{23}+2\cdot a_{33}\\ 0\cdot a_{11}+1\cdot a_{21}+0\cdot a_{31} & 0\cdot a_{12}+1\cdot a_{22}+0\cdot a_{32} & 0\cdot a_{13}+1\cdot a_{23}+0\cdot a_{33}\\ 0\cdot a_{11}+0\cdot a_{21}+1\cdot a_{31} & 0\cdot a_{12}+0\cdot a_{22}+1\cdot a_{32} & 0\cdot a_{13}+0\cdot a_{23}+1\cdot a_{33}\end{bmatrix}
$$

$$
= \begin{bmatrix}a_{11}+2a_{31} & a_{12}+2a_{32} & a_{13}+2a_{33}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}
$$

Matrices A and B are **row-equivalent** if A can be transformed to B by a finite sequence of elementary row operations.

Since row operations may be implemented by multiplying by elementary matrices, A and B are row-equivalent if and only if there are elementary matrices $E_1, \dots, E_n$ such that

$$E_1 \cdots E_n A = B$$

D Gauss–Jordan elimination

The **Gauss–Jordan elimination** is an algorithm to solve a system of linear equations by representing it as an augmented matrix, reducing it using row operations, and expressing the system in reduced row echelon form to find the values of the variables. We will see that solving a system of linear equations is closely related to finding the inverse of the matrix. We will see that Gauss–Jordan elimination can be used also for finding the inverse of a matrix.

First we need to define several new concepts. A matrix is in **row echelon form** if:

(i) All rows consisting of only zeroes are at the bottom.

(ii) The leading coefficient of a nonzero row is always *strictly to the right* of the leading coefficient of the row above it.

These two conditions imply that all entries in a column below a leading coefficient are zeros.

Example 10. The following two matrices are in row echelon form:

$$
\begin{bmatrix}1 & 1 & 3\\ 0 & -5 & -5\\ 0 & 0 & 0\end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix}1 & a_0 & a_1 & a_2 & a_3\\ 0 & 0 & 2 & a_4 & a_5\\ 0 & 0 & 0 & 1 & a_6\end{bmatrix}
$$

A matrix which is in row echelon form can be viewed as a "step matrix". The entries in the lower-left corner of the matrix are zeros, and the shape of the leading non-zero elements forms a shape of steps.

A matrix is in **reduced row echelon form** if it satisfies the following conditions:

- It is in row echelon form.
- The leading entry in each nonzero row is a 1 (called a leading 1).
- Each column containing a leading 1 has zeros in all its other entries.

The row reduced echelon form of a matrix is unique and every matrix can be always transformed to a reduced row echelon form.

Example 11. This matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & a_1 & 0 & b_1 \\ 0 & 1 & a_2 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{bmatrix}$$

Let us consider the following **system of equations**:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Here $x_1, x_2, \ldots, x_n$ are the unknown **variables**, $a_{11}, a_{12}, \ldots, a_{nn}$ are the **coefficients** of the system, and $b_1, b_2, \ldots, b_n$ are the constant terms. Our task is to find a **solution** of the linear system is an assignment of values to the variables $x_1, x_2, \ldots, x_n$ such that each of the equations is satisfied.

Let A be a matrix formed from the coefficients a_{ij} of the above system, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

This is called the **coefficient matrix** of the system.

We set $\mathbf{x}$ to be the column matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

consisting of all the unknowns. This can be also denoted

$$\mathbf{x} = [x_1, \ldots, x_n]^{\mathrm{T}}.$$

Note that we denote column matrices by **bold** so that the are not mixed with variables of numbers. We also set

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

that is, $\mathbf{b} = [b_1, \ldots, b_n]^{\mathrm{T}}$. Now the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

can be written in matrix form:

$$A\mathbf{x} = \mathbf{b}.$$

Please check that this really holds by doing the $A\mathbf{x}$ multiplication!

If A has an inverse A^{-1}, we can multiply from left the both sides of the equation $A\mathbf{x} = \mathbf{b}$ by A^{-1} :

$$A\mathbf{x} = b \iff A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \iff I\mathbf{x} = A^{-1}\mathbf{b} \iff \mathbf{x} = A^{-1}\mathbf{b}.$$

Knowing the inverse of the matrix solves the systems of linear equations related to that matrix.
We will also see that solving a system of equations produces the inverse of a matrix as a “side product”.

Finally, we will consider the Gauss-Jordan elimination method. It is an algorithm and can easily be programmed to solve a system of linear equations. The main idea of Gauss-Jordan Elimination is:

- To represent a system of linear equations in an augmented matrix form.
- Performing the row operations on it until the reduced row echelon form is achieved. Recall that the elementary row operations can be performed by multiplying with *elementary matrices* from left.

- Lastly, we can easily recognize the solutions from the reduced row echelon form

Example 12. In this simple example, we show how the Gauss-Jordan elimination works. Consider a system of equations:

$$\begin{array}{rcrl} x & + & y & = 3 \\ 3x & - & 2y & = 4 \end{array}$$

Let us first form the coefficient matrix of the system:

$$\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

Note that the first column contains the coefficients of the first unknown x and the second column contains the coefficients of the second unknown y. Next, the coefficient matrix is augmented by writing the constants that appear on the right-hand sides of the equations as an additional column:

$$\begin{bmatrix} 1 & 1 & 3 \\ 3 & -2 & 4 \end{bmatrix}$$

This is called the **augmented matrix**, and each row corresponds to an equation in the given system. Note that the first row, $r_1 = [1, 1, 3]$, corresponds to the first equation, $1x + 1y = 3$, and the second row, $r_2 = [3, -2, 4]$, corresponds to the second equation, $3x - 2y = 4$.

Next we use elementary row operations to transform the matrix to the the reduced row echelon form:

$$\begin{bmatrix} 1 & 1 & 3 \\ 3 & -2 & 4 \end{bmatrix} \overset{(1)}{\Rightarrow} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -5 & -5 \end{bmatrix} \overset{(2)}{\Rightarrow} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \overset{(3)}{\Rightarrow} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Here are the steps presented as matrix multiplications:

(1) Multiply the first row by -3 and add it to the second row:

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 3 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 3-3 & -3-2 & -9+4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -5 & -5 \end{bmatrix}$$

(2) Divide the second row by -5 ($=$ multiply with $-\frac{1}{5}$):

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -5 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -5 \cdot -\frac{1}{5} & -5 \cdot -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

(3) Multiply the second row by -1 and add it to the first row:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1-0 & 1-1 & 3-1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Because the first column corresponds to x and the second variable corresponds to y, we get the solution $x = 2$ and $y = 1$. What is also important is that we have now *transformed* A to an identity matrix.

Note that we need the fact that the elementary row operations preserve the solutions of a linear system of equations. Swapping rows is just changing the order of the equations begin considered, which certainly should not alter the solutions. Scalar multiplication is just multiplying the equation by the same number on both sides, which does not change the solution(s) of the equation. It is also true that if two equations share a common solution, adding one to the other preserves the solution. This is not proved in this course.

Example 13. This system of equations has three variable and three equations.

$$\begin{aligned} x-\ y+\ z &= 8 \\ 2x+3y-\ z &= -2 \\ 3x-2y-9z &= 9 \end{aligned}$$

Let us try to solve that system by using Gauss-Jordan elimination. First, we write the augmented matrix:

$$\begin{bmatrix} \boxed{1} & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{bmatrix}$$

We transform it to echelon form such that the leading coefficients are 1's. Note that in this example $\mathbf{R}_1$ denotes the first for of the matrix, $\mathbf{R}_2$ is the second row and $\mathbf{R}_3$ is the third one.

Let us consider the element in the upper left corner marked inside a box. It is already in its correct position. We aim to eliminate (reduce to "0") all the elements below it. This is done by adding multiples of R_1 to the other rows.

$\mathbf{-2R_1 + R_2 \to R_2}$:

$$\begin{bmatrix} \boxed{1} & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 3 & -2 & -9 & 9 \end{bmatrix}$$

Now the second row contains 0 in the first column.

$\mathbf{-3R_1 + R_3 \to R_3}$:

$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{bmatrix}$$

Now also the third row contains 0 in the first column.

Interchange R_2 **and** R_3 (because in row 3 the element in second column is 1):

$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & \boxed{1} & -12 & -15 \\ 0 & 5 & -3 & -18 \end{bmatrix}$$

Next we consider the leading 1 in second row (in the box) and try to eliminate the element below it:

$\mathbf{-5R_2 + R_3 \to R_3}$:

$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{bmatrix}$$

$\mathbf{-\frac{1}{57}R_3 \to R_3}$ (to have 1 as the leading coefficient):

$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

This matrix is in row echelon form such that the leading coefficients are 1. It needs to be transformed to the *reduced row echelon form.* This is done as follows:

$\mathbf{12R_3 + R_2 \to R_2}$:

$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\mathbf{1R_2 + R_1 \to R_1}$:

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\mathbf{-1R_2 + R_1 \to R_1}$:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

This is now in *reduced row echelon form.* Note also that we have now transformed A into identity matrix (three first columns)! From this form we see that the system of equations has the solution:

$$x = 4, \; y = -3, \; z = 1.$$

You may check that these numbers satisfy all the equations, for instance, the second equation $2x + 3y - z = -2$ of the system is satisfied, because

$$2 \cdot 4 + 3 \cdot (-3) - 1 = 8 - 9 - 1 = -2.$$

E Finding the inverse of a matrix

In this section, we show theoretically how finding the inverse of the matrix and solving a system of linear equations can be seen as equivalent tasks!

Theorem 14. *Let A be an $n \times n$ matrix. The following are equivalent:*

(a) *A is row equivalent to I_n.*

(b) *A is a product of elementary matrices.*

(c) *A is invertible.*

(d) *The system $A\mathbf{x} = \mathbf{0}$ of n unknowns $\mathbf{x} = [x_1, x_2, \dots, x_n]^{\mathrm{T}}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, where $\mathbf{0} = [0, 0, \dots, 0]^{\mathrm{T}}$.*

Proof. We prove (a)⇒(b), (b)⇒(c), (c)⇒(d) and (d)⇒(a).

(a)⇒(b): Assume that (a) holds. Let $E_1, \dots, E_p$ be elementary matrices which row reduce A to I, that is,

$$E_1 \cdots E_p A = I_n.$$

Then

$$(E_1^{-1} E_1) E_2 \cdots E_p A = E_1^{-1} I = E_1^{-1} \quad \text{and} \quad E_2 \cdots E_p A = E_1^{-1}.$$

Similarly,

$$E_3 \cdots E_p A = E_2^{-1} E_1^{-1}.$$

Finally we get:

$$A = E_p^{-1} \cdots E_1^{-1}.$$

Since the inverse of an elementary matrix is an elementary matrix, A is a product of elementary matrices. So (b) holds.

(b)⇒(c): Suppose that (b) holds. Write A as a product of elementary matrices: $A = F_1 \cdots F_q$. Now

$$F_1 \cdots F_q \cdot F_q^{-1} \cdots F_1^{-1} = I$$

and

$$F_q^{-1} \cdots F_1^{-1} \cdot F_1 \cdots F_q = I.$$

Hence,

$$A^{-1} = F_q^{-1} \cdots F_1^{-1}.$$

This means that A is invertible and (c) holds.

(c)⇒(d): Suppose that (c) holds, that is, A is invertible. If we multiply $A\mathbf{x} = \mathbf{0}$ both sides by A^{-1}, we get:

$$\mathbf{x} = A^{-1} A\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0},$$

that is, $\mathbf{0}$ is the one and only solution to the system.

(d)$\Rightarrow$(a): Assume that the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. If $A = [a_{ij}]_{n \times n}$, the augmented matrix is:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix}$$

This means that using Gauss–Jordan elimination, we can obtain the only possible solution $x_1 = 0, x_2 = 0, \ldots, x_n = 0$:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Note that the first n columns form the identity matrix I_n. This means that there is a sequence of row operations $E_1, \ldots, E_n$, which transforms A to the identity I_n. Thus, A is row equivalent to I_n. $\square$

Example 15. The matrix

$$A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$$

is not invertible. There are several ways to show this, but let us consider the system $A\mathbf{x} = \mathbf{0}$ of equations, that is,

$$\begin{aligned} 2x_1 - 4x_2 &= 0 \\ x_1 - 2x_2 &= 0 \end{aligned}$$

For instance, $x_1 = 2$ and $x_2 = 1$ satisfies both the equations. This means that is has more solutions than $\mathbf{x} = \mathbf{0}$. Theorem 14 implies that A is not invertible. In fact, we can select x_2 freely and set $x_1 = 2x_2$. This means that the system of equations has an infinite number of solutions.

Note also that

$$A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$$

cannot be row reduced to I_2. We can transform A to the reduced row echelon form:

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

But it appears that this matrix can be never transformed to the identity matrix I_2 by elementary row operations.

Finally, let us consider an example showing how we can find the inverse of a matrix.

Example 16. If A is invertible, Theorem 14 says that A can be written as a product of elementary matrices. We row reduce A to the identity (similarly as in Gauss-Jordan elimination) and write each row operation as an elementary matrix. For instance, if the used matrices are E_1, E_2, E_3 and E_4 in this order, then

$$A^{-1} = E_4 E_3 E_2 E_1.$$

For example, let

$$A = \begin{bmatrix} 2 & -4 \\ -2 & 3 \end{bmatrix}$$

We row reduce A to I:

$\mathbf{\frac{1}{2}R_1 \to R_1}$:

$$\begin{bmatrix} 2 & -4 \\ -2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$$

The corresponding elementary matrix is:

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$\mathbf{2R_1 + R_2 \to R_2}$:

$$\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$

The corresponding elementary matrix is:

$$E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$\mathbf{-1R_2 \to R_2}$:

$$\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

The corresponding elementary matrix is:

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\mathbf{R_1 + 2R_2 \to R_1}$:

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The corresponding elementary matrix is:

$$E_4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

We have been able to row reduce A to the identity, that is,

$$E_4 E_3 E_2 E_1 A = 1$$

But this means that

$$A^{-1} = E_4 E_3 E_2 E_1$$

Now

$$E_2 E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 1 \end{bmatrix},$$

$$E_3 \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & -1 \end{bmatrix},$$

$$E_4 \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1\frac{1}{2} & -2 \\ -1 & -1 \end{bmatrix}.$$

The last one is the inverse matrix of A because

$$\begin{bmatrix} 2 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1\frac{1}{2} & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -1\frac{1}{2} & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finally, note that matrix

$$\begin{bmatrix} -1\frac{1}{2} & -2 \\ -1 & -1 \end{bmatrix}$$

could be obtained starting from the identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and by doing the elementary row operations corresponding to the matrices $E_1, \ldots, E_4$ (please check yourself):

- $\mathbf{\frac{1}{2}R_1 \to R_1}$
- $\mathbf{2R_1 + R_2 \to R_2}$
- $\mathbf{-1R_2 \to R_2}$

- $\mathbf{R_1 + 2R_2 \to R_1}$

We can now prove the following corollary of Theorem 14.

Corollary 17. *Let $A = [a_{ij}]_{n\times n}$. The following are equivalent:*

(a) *A is invertible.*

(b) *For any $\mathbf{b} = [b_1, \dots, b_n]^{\mathrm{T}}$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.*

Proof. (a)⇒(b): Suppose that A is invertible. As we have already noted, we can multiply the both sides of $A\mathbf{x} = \mathbf{b}$ from left by A^{-1}. We obtain

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b},$$

which means that

$$\mathbf{x} = A^{-1}\mathbf{b}$$

is a solution.

Suppose that $A\mathbf{x} = \mathbf{b}$ has two solutions, which are denoted by $\mathbf{x_1}$ and $\mathbf{x_2}$. Note that matrix product is distributive over sum, that is, $A(B+C) = AB + AC$ (unfortunately, not proved in this course). We have that

$$A(\mathbf{x_1} - \mathbf{x_2}) = A\mathbf{x_1} - A\mathbf{x_2} = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

This means that $\mathbf{x_1} - \mathbf{x_2}$ is a solution to $A\mathbf{x} = \mathbf{0}$. Because A is invertible, by Theorem 14, the only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = \mathbf{0}$. We have that $\mathbf{x_1} - \mathbf{x_2} = \mathbf{0}$ yielding $\mathbf{x_1} = \mathbf{x_2}$. This means that $A^{-1}\mathbf{b}$ is the only solution to $A\mathbf{x} = \mathbf{b}$.

(b)⇒(a): Suppose $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$. In particular, for $\mathbf{b} = \mathbf{0}$, the system $A\mathbf{x} = 0$ has a unique solution (which is $\mathbf{x} = \mathbf{0}$). By Theorem 14, A is invertible. □

We end this section by showing that two square matrices A and B are inverses requires only half of the work.

Proposition 18. *If A and B are $n \times n$ matrices and $AB = I$, then $A = B^{-1}$ and $BA = I$.*

Proof. Suppose that $AB = I$. The system

$$B\mathbf{x} = \mathbf{0}$$

has $\mathbf{x} = \mathbf{0}$ as a solution. Suppose for contradiction that y is another solution, that is,

$$B\mathbf{y} = \mathbf{0}$$

Let us multiply both sides by A and simplify:

$$AB\mathbf{y} = A\mathbf{0} \iff I\mathbf{y} = 0 \iff \mathbf{y} = 0.$$

This means that $\mathbf{y} = \mathbf{x}$ and $\mathbf{x}$ is the only solution. Therefore, by Theorem 14, B is invertible. We have that

$$AB = I \Rightarrow ABB^{-1} = IB^{-1} \Rightarrow AI = B^{-1} \Rightarrow A = B^{-1}.$$

Moreover,

$$BA = BB^{-1} = I. \qquad \square$$

F Boolean matrices and relations

A **Boolean matrix** is a matrix with entries from the Boolean domain $\{0, 1\}$. Such a matrix can be used to represent a binary relation between a pair of finite sets. If R is a binary relation between finite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$, then R can be represented by the Boolean matrix M whose entries are defined by:

$$M_{ij} = \begin{cases} 1 & (x_i, y_j) \in R \\ 0 & (x_i, y_j) \notin R \end{cases}$$

Example 19. Let us consider the 'divides' relation $|$ on the set $\{1, 2, 3, 4\}$. For example, $2|4$ holds, because 2 divides 4, but $3|4$ does not hold because $4/3$ has a remainder of 1. The following set is the set of pairs for which the relation $|$ holds:

$$\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

The corresponding representation as a Boolean matrix is:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The product of Boolean matrices works very similarly as the matrix multiplication. Let A be an $m \times k$ matrix and B be a $k \times n$ matrix, and that both matrices are zero-one matrices. Then the Boolean product of $A = [a_{ij}]_{m\times k}$ and $B = [b_{ij}]_{k\times n}$, denoted by $A \circ B$, is the $m \times n$ matrix $C = [c_{ij}]$ such that

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ip} \wedge b_{pj}).$$

This Boolean product form is very similar to the matrix multiplication formula, except now we replace $\cdot$ with $\wedge$ and $+$ with $\vee$. In addition, the product of $A \circ B$ is undefined if the number of columns of A is not the same as the number of rows of B. Suppose A is an $m \times k$ matrix and B is $l \times n$ matrix, then $A \circ B$ is defined only if $k = l$ and similarly $B \circ A$ is defined only if $n = m$.

If the relations R and S are represented by their matrices M_R and M_S, then the composition $R \circ S$ of relations is represented by the matrix product $M_R \circ M_S$.

Example 20. Let $A = \{a, b, c\}$, $B = \{1, 2\}$, and $C = \{x, y, z\}$. Let R be a relation from A to B such that

$$R = \{(a, 1), (b, 2), (c, 1), (c, 2).$$

Then, the matrix M_R is a 3×2 matrix such that

$$M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let S be a relation from B to C defined so that

$$S = \{(1, x), (1, z), (2, y), (2, z)\}.$$

Its matrix is a 2×3-matrix

$$M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now $M_R \circ M_S$ is a 3×3-matrix

$$M_R \circ M_S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Consider for example the element in box. It is computed as

$$(0 \wedge 0) \vee (1 \wedge 1) = 0 \vee 1 = 1.$$

On the other hand, $R \circ S$ is a relation from A to C such that

$$R \circ S = \{(a, x), (a, z), (b, y), (b, z), (c, x), (c, y), (c, z)\}.$$

Its matrix $M_{R \circ S}$ is a 3×3 matrix

$$M_{R \circ S} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Note that $M_R \circ M_S = M_{R \circ S}$.

We end this course by noting that the transpose M_R^{T} is the matrix of the inverse of the relation R.

Example 21. Let $X = \{a, b, c\}$ and $Y = \{1, 2\}$. Let the relation $R \subseteq X \times Y$ be such that

$$R = \{(a, 1), (a, 2), (b, 1), (c, 2)\}.$$

The Boolean matrix of R is

$$M_R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The relation R^{-1} is a relation from Y to X and its matrix is

$$M_{R^{-1}} = {M_R}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$