# Conditional Probability

Suppose that in a certain city, 30% of the days are rainy, that is,

$$P(\mathbf{Rain}) = \mathbf{0.3}$$

The probability that it rains given that it is cloudy should bigger, like:

$$P(\mathbf{Rain}|\mathbf{Cloudy}) = \mathbf{0.9}$$

**Conditional probability:** the probability of *A* given *B*:

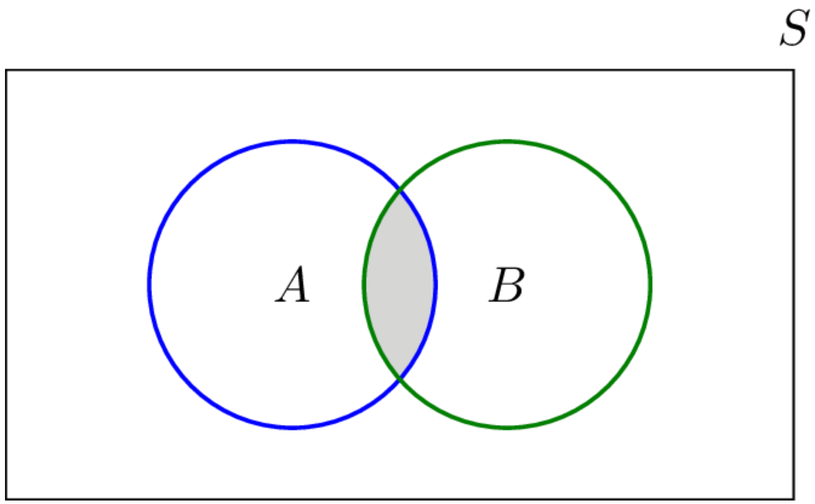
$$P(A|B)$$

"Probability of *A* in the case *B* is already happened"

# Conditional probability

**Definition.** The **conditional probability** , the probability that *A* occurs given that *B* has occurred is defined by:

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ when } P(B) > 0.}$$

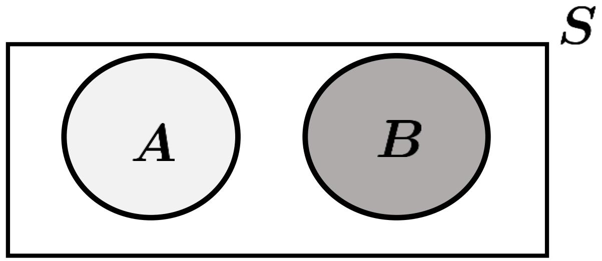


# Conditional Probability

Special cases:

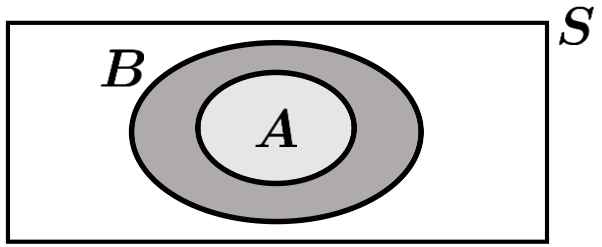
1) $A$ and $B$ are disjoint:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0.$$




2) $A \subseteq B$:

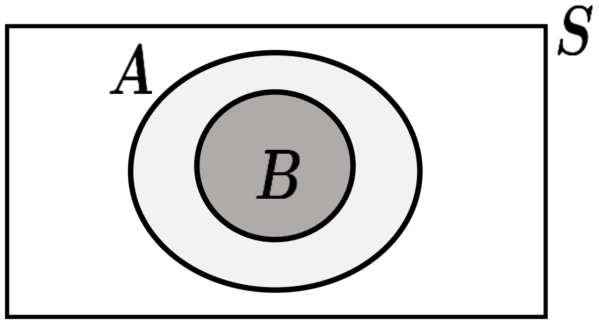
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$



# Conditional Probability

3) $B \subseteq A$:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$




4) There are two ways to express the intersection:

$$P(A \cap B) = P(A)P(A|B) = P(B)P(B|A)$$

# EXAMPLE

- Family that has two children

# Independence

Let us define the following events:

- *A* = “it will rain tomorrow”
- *B* = “I toss a coin and it lands heads up”

We can assume that $P(A) = 0.3$ and $P(B) = 0.5$

What is $P(A|B)$?

It should clear that $P(A\ |B) = P(A) = 0.3$

The result of my coin toss does not have anything to do with tomorrow’s weather.

No matter *B* happens or not, the probability of *A* is affected

This is an example of *independent events.*

# Independence

**Definition.** Two events A and B are **independent** if

$$P(A \cap B) = P(A)P(B)$$

**Example**. If a fair coin is flipped two times, the flips are **not** a affecting to each other, so they are *independent*. Let

- A = “first throw is heads”, $P(A) = 0.5$
- B = “second throw is heads”, $P(B) = 0.5$
- C=“both flips are heads”.

Then,

$$P(C) = P(A)P(B) = 0.5 \cdot 0.5 = 0.25$$

# Independence

**Proposition**. The following are equivalent:

(a) $P(A|B) = P(A)$

(b) $P(A \cap B) = P(A)P(B)$

(c) $P(B|A) = P(B)$

*Proof*: We prove (a) => (b) and (b) => (a). The equivalence of (b) and (c) can be proved similarly. This means that all cases are equivalent.