Reasoning and predicate logic

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Deductive reasoning

- Deductive reasoning is a logical process where the end result (conclusion) is reached by connecting the premises
- Reasoning that is done in natural language is called natural reasoning
- Natural language is too imprecise for more complicated reasoning, so reasoning tasks are often formalized to symbolic form:
  - Premises are marked as $p_1, p_2, p_3, \dots, p_n$
  - Based on the premises we can make a conclusion q if and only if the argument $p_1 \land p_2 \land p_3 \land \cdots \land p_n \Rightarrow q$ is a tautology
- This is called formal reasoning

Deductive reasoning

- Often the reasoning process is marked in such a way that we write the premises one below another and the conclusion under a line
- Symbol ”$\therefore$” means ”therefore”

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ \underline{p_n} \\ \therefore q \end{array}$$

- Good logical reasoning ”preserves truth”
  - If the premises are true, also the conclusion is true
  - Therefore if the conclusion is false, premises are wrong

Semantic vs. syntactic reasoning

- The reasoning process (i.e. proving the argument true or false) can be done in semantic or syntactic way
- Semantic reasoning = examine the truth values using a truth table and see if the argument ends up to be a tautology
  - Rather bulletproof method
- Syntactic reasoning = simplify the argument using already known tautologies
  - Quicker, if we make good decision in simplification
- Syntactic reasoning is difficult to automatize, because we can't know which tautology we should substitute
  - Trial-and-error procedure needs to be able to recover from bad decisions
- More on this a bit later…

Rules of inference

- In mathematical proofs, certain tautologies are often used in simplifying the argument
- These tautologies are called rules of inference
- Many of these have their own names:

Rule of inference	Name
$(p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p)$	Material equivalence
$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$	Transposition
$(p \Rightarrow q) \wedge p \Rightarrow q$	Modus ponens
$(p \Rightarrow q) \wedge \neg q \Rightarrow \neg p$	Modus tollens
$p \wedge (\neg q \Rightarrow \neg p) \Rightarrow q$	Indirect proof
$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$	(Hypothetical) syllogism

Material equivalence

- If p leads to q and q leads to p, we can construct a truth table and notice that p and q must be equivalent
- Example: let's examine a continuous function f
  - p = "the derivative of f is never zero"
  - q = "f is truly monotonic"
  - Implication can be written in both directions and both are equally true
  - Therefore, p and q are equivalent

$$(p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p)$$

Transposition

- Transposition means proof through negations: if the original argument is "if p, then q", a logically equivalent argument is "if not q, then not p"

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

- This proof is rather common, because it is generally work-heavy to prove that implications are true
- Example: prove that for all integers, if n^2 is divisible by 3, also n is divisible by 3
  - Define p = "n^2 is divisible by 3" and q = "n is divisible by 3"
  - Then the negations are $\neg q$ = "n is not divisible by 3" and $\neg p$ = "n^2 is not divisible by 3"

Transposition

- Integers that are “not divisible by 3” are either of form $n_1 = 3k + 1$ or $n_2 = 3k + 2$. Let’s test whether either of these will fulfill the divisibility of n^2:

$$n_1^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1.$$

Not divisible by 3; there would be a remainder 1/3

$$n_2^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k) + 4.$$

Not divisible by 3; there would be a remainder 4/3

- Hence, $\neg q \Rightarrow \neg p$ is always true
- Therefore according to transposition also the original argument $p \Rightarrow q$ is always true

Modus ponens

- Very common deductive chain is composed of the following premises:
  - If p, then q
  - We know for sure that p is true
- If both premises are true at the same time, we can be sure that q is true
- This deductive chain is called "modus ponendo ponens", which is often simply written as "modus ponens"

$$(p \Rightarrow q) \wedge p \Rightarrow q$$

Modus tollens

- "Modus tollendo tollens" or "modus tollens" for short is a close relative of the former; this time we begin with the following premises:
  - If p, then q
  - We know, that q is false
- If both premises are true, the only possibility is that p is false

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

$$(p \Rightarrow q) \land \neg q \Rightarrow \neg p$$

Indirect proof

- Rule of indirect proof is basically a combination of modus ponens and transposition:

$$(p \Rightarrow q) \land p \Rightarrow q$$ Modus ponens

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$ Transposition

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$$(\neg q \Rightarrow \neg p) \land p \Rightarrow q$$

Indirect proof

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$$(p \Rightarrow q) \land p \Rightarrow q$$ Modus ponens

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$ Transposition

$$(\neg q \Rightarrow \neg p) \land p \Rightarrow q$$ Commutation 2

$$p \land (\neg q \Rightarrow \neg p) \Rightarrow q$$

Hypothetical syllogism

- Hypothetical syllogism (or just ”syllogism”) provides a ”shortcut” through several consecutive implications:
  - ”If p, then q”
  - ”If q, then r”
- So, if both premises are true, the only logical conclusion is ”if p, then r”
- Used often especially in higher-order logic

$$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$$

Predicates

- Previously we examined propositional logic, where the truth value of a (closed) proposition was easy to comprehend: when expressed, it is unconditionally either true or false
  - For example, ”it’s raining”: check out the window and find out whether the proposition is true or false
- Often we have situations, where the truth value of an expression is dependent on some quantified variable(s)
- This kind of an (open) expression is called a predicate
- Respectively, logic that considers predicates is called predicate logic or first-order logic

Definition of variable

- In predicate logic, the domain of the variable plays a very important role
- For example predicate p(x)="$\sqrt{x+1} < 3$":
  - Domain must be limited: $x \geq -1$
  - If $x \in \mathbb{R}$, predicate is true when $x \in [-1,8[$
  - If $x \in \mathbb{Z}$, predicate is true when predicates p(-1), p(0), …, p(7) are true
  - If $x \in \mathbb{N}$, predicates p(0), p(1), …, p(7) are true

Truth value of a predicate

- When the truth value of a predicate is dependent on the variable, it is extremely important to be precise about when the conditions are met
- Transformation from natural language to predicate logic language is sometimes challenging
- Example 1: ”Crows have feathers”
  - Interpretation a: all crows always have feathers
  - Interpretation b: some crows have feathers
- Example 2: ”Products are on sale”
  - Interpretation a: all products are on sale
  - Interpretation b: some products are on sale

Quantifiers

- In order to be able to define exactly what we mean, we use quantifiers
- Quantifies help to unambiguously define the variable and therefore transform an open expression to a closed one
  - The variable is no longer free but tied to a certain domain
  - Therefore we are able to define a clear truth value for the expression
- There are two types of quantifiers:
  - Universal quantifier
  - Existential quantifier

Universal quantifier ∀

- Self-evidently ∀x means ”for all x”
- In natural language, common equivalents include
  - ”for every x”
  - ”any value of x”
- Example use: expression

$$\forall \boldsymbol{x} \in \boldsymbol{A} : \boldsymbol{p}(\boldsymbol{x})$$

- Interpretation: ”All members of A have the property p”
- …or ”Property p(x) is valid always when $x \in A$”

Existential quantifier ∃

- Also rather self-evident: means that ”there exist a value/values, which fulfill condition ___”
- Equivalents in natural language also include:
  - ”for some values”
  - ”for at least one value”
- Example use: expression

$$\exists \boldsymbol{x} \in \boldsymbol{A} \colon \boldsymbol{p}(\boldsymbol{x})$$

- Interpretation: ”In set A there exists at least one member that has property p”
- …or ”p is valid for at least one member of A”

Examples of predicate logic

- Let's examine the truth values of some expressions:

$$\forall x \in \mathbf{R}: x^2 \geq 0$$

  - "The square of any real number is non-negative"
  - Expression is true

$$\forall \boldsymbol{x} \in \mathbb{R}: \boldsymbol{x}^{\mathbf{2}} - \mathbf{1} > \mathbf{0}$$

  - " Any real number squared minus 1 is positive"
  - Expression is false, which can be proven by a counterexample: substitute x = 1, when we get $1^2 - 1 > 0$ which simplifies to $0 > 0$, which is undeniably false.
  - In general, all expressions that include a universal quantifier are easier to prove false by using a counterexample

Examples of predicate logic

$$\exists x \in \mathbf{R}: x^2 - 5x + 6 = 0$$

- "There exists at least one real number which fulfills the equation $x^2 - 5x + 6 = 0$"
- Using a solution formula for a 2nd degree equation we can show that this equation has two real solutions: x = 2 and x = 3
- Hence the expression is true
- In general, expressions that contain an existential quantifier are easier to prove true, since it only takes one example

Reasoning in predicate logic

- In predicate logic, truth tables are not valid anymore (because truth values of expressions depend on values of the variables)
- Hence semantic reasoning via truth tables doesn't work
- We'll have to resort to syntactic reasoning
  - Difficult to automatize, remember?
- Technique called resolution proof can be used:
  - Convert natural language statements to predicate logic
  - Convert the statements into conjunctive normal form (CNF)
  - Negate the statement to prove
  - Perform substitutions (resolution tree might be helpful)
  - Good explanation here: https://www.javatpoint.com/ai-resolution-in-first-order-logic
  - This will be used in course "Foundations of Artificial Intelligence"

Socrates-example

- One traditional example of a logical deduction problem that requires predicate logic is so called ”mortality of Socrates” (in reality not expressed by Socrates nor even Aristotle):
  - All humans are mortals.
  - Socrates is a human.
  - Therefore, Socrates is mortal.

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  - p = ”all humans are mortals”
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  - This isn’t a tautology, so propositional logic can not prove Socrates to be mortal

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- Formalize in predicate logic, then:
  - H(x) = ”x is human”
  - M(x) = ”x is mortal”
  - s = ”Socrates”

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  - H(x) = ”x is human”
  - M(x) = ”x is mortal”
  - s = ”Socrates”

$$\begin{array}{c} \forall x(H(x) \Rightarrow M(x)) \\ H(s) \\ \hline \therefore M(s) \end{array}$$

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  - H(x) = ”x is human”
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  - s = ”Socrates”

$$\frac{\begin{array}{c}\forall x(H(x) \Rightarrow M(x))\\ H(s)\end{array}}{\therefore M(s)}$$

  - Now this can be proven true! (Modus ponens)

Several quantifiers

- It is also possible to use more than one quantifier
- In practice, the need for this springs from the predicate having several variables
- Example: which of the following predicates are true and which are false?

$$
\begin{aligned}
&(1) \qquad \forall x \in \mathbb{R} : \forall y \in \mathbb{R} : x + y = 0, \\
&(2) \qquad \exists x \in \mathbb{R} : \exists y \in \mathbb{R} : x + y = 0, \\
&(3) \qquad \forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x + y = 0, \\
&(4) \qquad \exists y \in \mathbb{R} : \forall x \in \mathbb{R} : x + y = 0.
\end{aligned}
$$

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$$(2) \quad \exists x \in \mathbb{R} : \exists y \in \mathbb{R} : x + y = 0,$$

$$(3) \quad \forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x + y = 0,$$

$$(4) \quad \exists y \in \mathbb{R} : \forall x \in \mathbb{R} : x + y = 0.$$

- Answer: 1 and 4 are false, 2 and 3 are true (notice the importance of order!)

Several quantifiers

- Explanations:
  - (1) All real numbers naturally can't fulfill condition $x + y = 0$. Proof by example: $x = 5$, $y = 2 \rightarrow 5 + 2 = 7 \neq 0$
  - (2) There exists at least one pair of real numbers which fulfill condition $x + y = 0$. True – for example $x = 2$, $y = -2$ $\rightarrow 2 + (-2) = 0$
  - (3) The predicate can be read: "For each real number x there exists some y so that $x + y = 0$". This is true, because we can always select $y = -x$.
  - (4) The predicate can be read: "There exists such a real number y that for all values of x, the condition $x + y = 0$ is true". This is false, because one y-value can't fulfill the condition for all x's.
- Note: In predicate (3) y can be dependent on x, but the y in predicate (4) is selected first and hence that one y-value should fulfill the condition for all values of x.

Connectives and quantifiers

- We can combine open expressions using connectives (just like we did with propositional logic)
- Open expressions can then be closed using quantifiers
- Example: open expression

$$x + 1 > 0 \wedge 2x - 3 < 0$$

- Let's close this using an existential quantifier. We get

$$\exists x \in \mathbb{R} : x + 1 > 0 \wedge 2x - 3 < 0$$

  - Is the expression true?
  - What about if we close the expression with a universal quantifier?

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- Let's close this using an existential quantifier. We get

$$\exists x \in \mathbb{R}: x + 1 > 0 \wedge 2x - 3 < 0$$

  - Is the expression true? Yes, because we can find at least one value for x which fulfills the condition; for example x = 0
  - What about if we close the expression with a universal quantifier? In this case the expression is false, because already the first condition is false when for example x = -3. Hence the expression can't be true for all real numbers.

Example: x loves y

- Express the following predicates in natural language.

$\forall x \forall y \; loves(x, y)$

$\forall x \exists y \; loves(x, y)$

$\exists y \forall x \; loves(x, y)$

$\exists x \forall y \; loves(x, y)$

$\forall y \exists x \; loves(x, y)$

Example: x loves y

- Express the following predicates in natural language.

  $\forall x \forall y \; loves(x, y)$

  - "Everybody loves everybody."

  $\forall x \exists y \; loves(x, y)$

  - "Everybody loves somebody."

  $\exists y \forall x \; loves(x, y)$

  - "Somebody is loved by everyone."

  $\exists x \forall y \; loves(x, y)$

  - "Somebody loves everybody."

  $\forall y \exists x \; loves(x, y)$

  - "Everybody is loved by somebody."

Thank you!