Recurrence relations

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Recursion formula of 1st order

- In prior lecture we derived a closed form formula for the recursion formula

$$p_{n+1} = 3p_n - 1$$

- Recursion formulas of this form are rather common – coefficient parameters naturally vary
- Could we possibly be able to derive a closed form formula for a general 1st order recursion formula – which, written in parametrized form, is

$$p_{n+1} = ap_n + b$$

- ...and even such a way that the initial value is left as a variable, so $p_0 = x$ like we did last time?

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$$p_2 = a(ax + b\ ) + b = a^2x + ab + b$$

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$$p_n = a^nx + b\left(\sum_{i=0}^{n-1} a^i\right)$$

Recursion formula of 1st order

- The latter term is a geometric sum, where the common ratio is q = a. Let's replace the sum notation by the formula for geometric sum, so we'll get

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$$= x + b(0) = x$$

- Works!

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$$p_{k+1} = ap_k + b = a\left(a^k x + b\left(\frac{1-a^k}{1-a}\right)\right) + b$$

Multiply a inside the brackets

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$$= a^{k+1} x + b\left(\frac{a-a^{k+1}}{1-a}\right) + b = a^{k+1} x + b\left(\frac{a-a^{k+1}}{1-a} + 1\right)$$

Take b as common multiple

Recursion formula of 1st order

- Expand the 1 by term (1-a), so that we get both terms in brackets to have the same denominator:

$$= a^{k+1}x + b\left(\frac{a - a^{k+1}}{1 - a} + \frac{1 - a}{1 - a}\right)$$

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- We managed to modify the left side to match the right side, so the claim is correct!

- Conclusion: the derived closed form formula is correct

Recurrence relations

- So, we managed to derive that the recursive formula

$$p_{n+1} = ap_n + b \qquad , p_0 = x$$

- …has a closed form solution

$$p_n = a^n x + b\left(\frac{1 - a^n}{1 - a}\right)$$

- Generally speaking, recursive formulae can be expressed in forms that are called recurrence relations
- For example, a recurrence relation that equals this recursive formula is (traditionally written using y)

$$y_{n+1} - ay_n = b$$

Recurrence relations of higher order

- We already derived a closed from formula (so, a solution) for a 1st order recursion formula
- We did this purely by using our own heuristic and proof by induction
- With higher order recurrence relations this method is quite work-heavy, so we'd need some handier tools
- Luckily, such tools exist!
- The solutions are based on the beforementioned nature of recurrence relations, where the order number *n* will appear in the exponent of the solution
- Let's start by examining a homogeneous recurrence relation of 2nd order

Homogeneous recurrence relation of 2nd order

- Homogeneous recurrence relation of 2nd order is of form

$$y_{n+2} + ay_{n+1} + by_n = 0$$

- "Homogeneous" = right side is zero (only y-terms)
- In principle, coefficients a and b could also be dependent on the order number n, but let's now consider only the most common case where a and b are constant coefficients

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- By understanding that this means $y_{n+1} = r^{n+1}$ and $y_{n+2} = r^{n+2}$ and by substituting these to the recurrence relation we get it to a form

$$r^{n+2} + ar^{n+1} + br^n = 0$$

Homogeneous recurrence relation of 2nd order

- Using exponent laws and by taking r^n as a common multiple we can modify the equation to form

$$r^2 r^n + a r r^n + b r^n = 0 \quad \Longrightarrow \quad r^n(r^2 + ar + b) = 0$$

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- Situation $r^n = 0$ corresponds to trivial solution $r = 0$, so the only “rational” solution can be found from the part in brackets; using a solution formula for a 2nd degree polynomial equation we can easily find out the roots of this equation – so, r_1 and r_2

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- This derivation actually applies for all homogeneous recurrence relations of nth order, so all recurrence relations of this form can be converted to such a polynomial equation (of nth order, respectively)
- The part in brackets is called a characteristic equation

$$r^2 + ar + b = 0$$

Homogeneous recurrence relation of 2nd order

- The total solution of a recurrence relation is formed by superposition from both solutions of the characteristic equation (r_1 and r_2) separately raised to a power of n:

$$y_n = r_1^n + r_2^n$$

- Because the recurrence relation is always satisfied (= the left side of the equation always gets a value 0) by these r-values, naturally the same happens for all their multiples, too:

$$y_n = c_1 r_1^n + c_2 r_2^n$$

- This form, where c_1 and c_2 are arbitrary constants, is the general solution of a homogeneous recurrence relation of 2nd order – that is, if the characteristic equation has two real roots

Solutions of a homogeneous recurrence relation of 2nd order

- We remember, that a 2nd degree equation had three possible outcomes regarding the number of solutions:
  - 2 real roots
  - 1 root (so called double root)
  - Complex roots (no real solution)
- Due to this nature also a homogeneous recurrence relation of 2nd order has three possible situations – depending on what kind of solutions we get from the characteristic equation:

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$$\begin{array}{ll} \boldsymbol{y_n = c_1 r_1^n + c_2 r_2^n} & \boldsymbol{if\ r_1 \neq r_2} \\ \boldsymbol{y_n = c_1 r^n + c_2 n r^n} & \boldsymbol{if\ r_1 = r_2 = r} \\ \boldsymbol{y_n = R^n(c_1 \cos(n\theta) + c_2 \sin(n\theta))} & \boldsymbol{if\ r_1 = \alpha + i\beta, r_2 = \alpha - i\beta} \end{array}$$

$$\boldsymbol{R = \sqrt{\alpha^2 + \beta^2}}$$

$$\begin{cases} if\ \alpha \neq 0, & \theta = \tan^{-1}\left({}^{\beta}/_{\alpha}\right) \\ if\ \alpha = 0, & \theta = {}^{\pi}/_{2} \end{cases}$$

Homogeneous recurrence relation of 2nd order; initial conditions

- So far we haven't taken into account the initial conditions y_0 and y_1, even though these naturally play a big role in the solution
- The effect of initial values will be considered via coefficients c_1 and c_2: these coefficients must be chosen in such a way that the initial conditions are met

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- So, in the end we have to solve these coefficients
  - Group of equations – for example, if 2 real solutions:

$$\begin{cases} y_0 = c_1 r_1^0 + c_2 r_2^0 \\ y_1 = c_1 r_1^1 + c_2 r_2^1 \end{cases}$$

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- Generally we can say the following:
  - Coefficients of the recurrence relation will define the shape of the solution
  - Initial conditions define the c-coefficients of solution terms

Homogeneous recurrence relation of nth order

- Previously presented solution methods works for all homogeneous recurrence relations with constant coefficients regardless of the order
- Therefore, we can solve a recurrence relation of any order like this
- In higher order relations we'll naturally have to solve a higher order characteristic equation
  - Analytical solutions can be cumbersome, unless we find easy solutions by experimentation and can use long division
  - No problem if we can use a calculator/computer
- Solutions are combined by superposition according to the previous table
  - For example, if 3rd order & characteristic equation has one double root and one single root: $r_1 = r_2$ and r_3

$$y_n = c_1 r_1^n + c_2 n r_1^n + c_3 r_3^n$$

Nonhomogeneous recurrence relation

- How about if the right hand side is not zero, but there exists a constant term d_n?
  - This term can either be an "actual" constant or it can depend on the order number n

$$y_{n+2} + ay_{n+1} + by_n = d_n$$

- The solution for such a recurrence relation is formed by combining the general solution of a corresponding homogeneous recurrence relation $y_{n,h}$ and a so called particular solution $y_{n,p}$ - by superposition, naturally:

$$y_n = y_{n,h} + y_{n,p}$$

- The solution $y_{n,h}$ we could solve using the beforementioned process, so now we just have to find the particular solution $y_{n,p}$

Nonhomogeneous recurrence relation

- The particular solution is solved using the method of undetermined coefficients
- Here we'll make an educated guess on what form the $y_{n,p}$ is going to be – based on the form of the nonhomogeneous part d_n - and add undetermined coefficient(s) in front of the terms
  - if d_n is of the same form as one of the solutions to the homogeneous equation, the $y_{n,p}$ must be multiplied by n
- When a suitable guess $y_{n,p}$ has been chosen, it will be substituted to the original recurrence relation
- If the $y_{n,p}$ choice was successful, the recurrence relation can be simplified to a form where we can solve the undetermined coefficient(s)

Recurrence relation: Example

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➡ $r_1 = \frac{2-4}{2} = \frac{-2}{2} = -1 \qquad r_2 = \frac{2+4}{2} = \frac{6}{2} = 3$

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$$-3A = 1$$

$$A = -\frac{1}{3}$$

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- Now when we've solved the undetermined coefficient A (which is hence no longer undetermined), we can write the particular solution:

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- Had we been given the initial values y_0 and y_1, we could substitute these to the solution and find out values for coefficients c_1 and c_2

Recurrence relations vs. difference equations

- In a recurrence relation we define the next term using the prior terms
- The corresponding equation could be written in such a way that we'd define the differences Δ of consecutive terms

$$\Delta(y_n) = y_{n+1} - y_n$$

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Combined

$$\Delta^2(y_n) = (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n) = y_{n+2} - 2y_{n+1} + y_n$$

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$$\Delta^2(y_n) = (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n) = y_{n+2} - 2y_{n+1} + y_n$$

- Equations written in this Δ-form are called difference equations*
- Difference equations are strongly linked to differential equations: they are actually discretized differential equations!

$$dx \longrightarrow \Delta x$$

*Terminology is a bit hazy, though: some authors speak of recurrence relations as difference equations

Difference equations

- Difference equations often arise in programming
  - Big-Oh complexity calculation for algorithms often leads to a difference equation (since the number of options is an integer, not a continuous variable)
- Also common in biology & geography
  - Migration & mixing of species
  - Same principles can be applied to economics (trickle-down economics models, globalization)
- A difference equation of higher order can be broken down to a group of first-order difference equations
  - Solving these is very similar to solving groups of differential equations
  - Matrix calculation provides good tools for this
- Close relationship often leads to people using more familiar differential equation models even though their variables would be of discrete nature

Thank you!