Decision analysis and expected utility hypothesis

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Decision analysis

- Especially in business (and naturally many other areas, too) it is common that we must answer the question “what should we do next?”
- When the decision process deals with quantifiable concepts in such a way that the goal can be expressed numerically (e.g., minimize losses/risk, maximize profits), it is natural that this problem is of mathematical nature
- Therefore, mathematics must be able to provide tools which can help to find an answer to the problem
- One branch of mathematics that deals with problems like this is *decision analysis*
  - Heavily connected to statistics and probability calculations

Decision tree

- Decision analysis can be made using multiple tools, but by far the most common (and easiest to implement) method is to employ a *decision tree*
  - Simple idea, graphical nature helps understanding
- A decision tree consists of a root and three different kinds of nodes, and it is drawn followingly:
  - *Root* (starting point) to left*, all possible scenarios branch and proceed from left to right
  - Nodes which represent choice situations (*decision nodes*) are drawn as squares; edges leaving these are *options*
  - Nodes which represent random outcomes (*chance nodes*) are drawn as circles; edges leaving these are *chances*
  - Nodes which are final outcomes of decision paths (*end nodes*) are drawn as triangles or left without a symbol with just the final *outcome value* written

*Some authors prefer top-to-bottom instead of left-to-right progression.

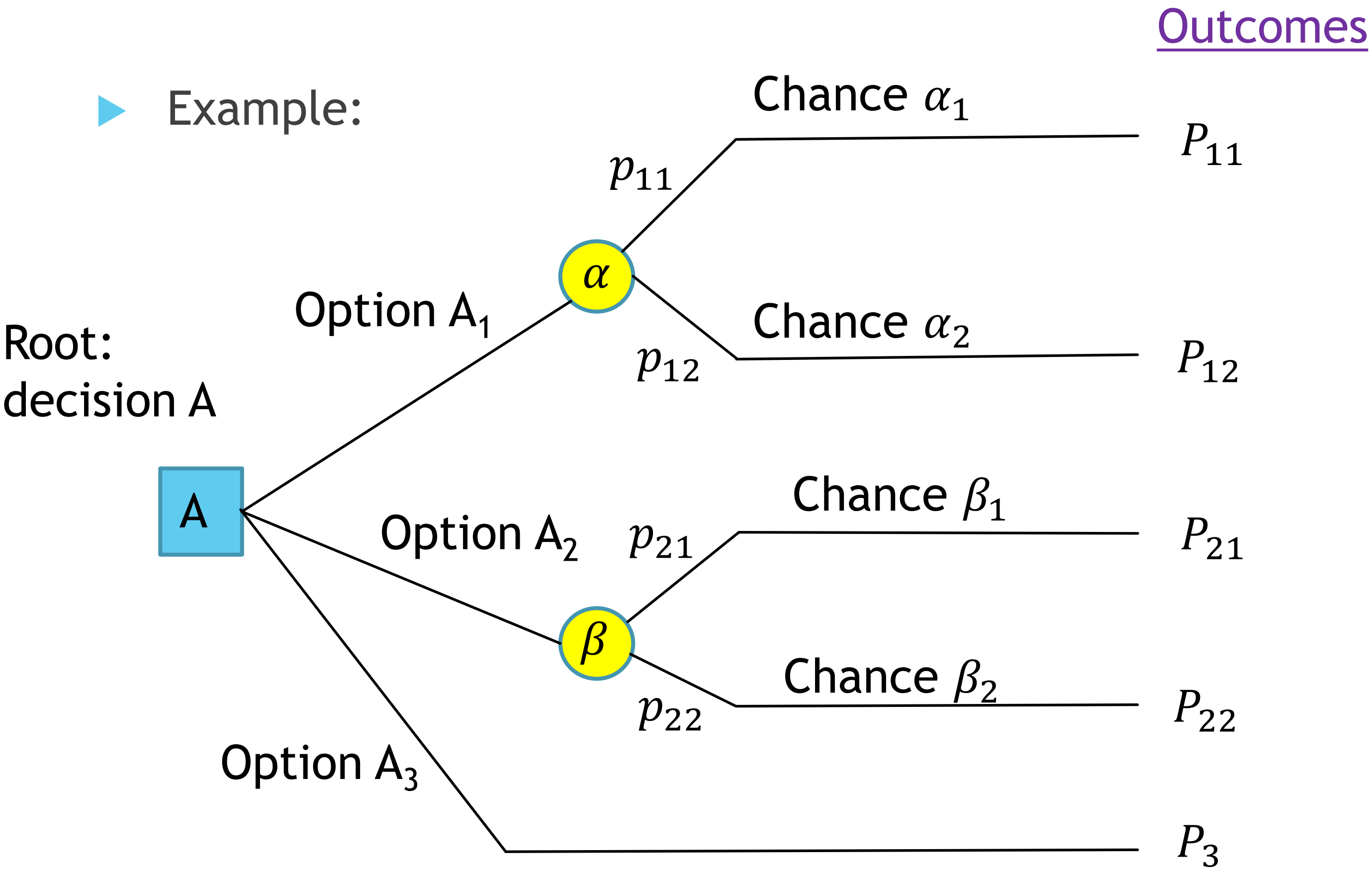
Decision tree

- Decision nodes are usually labeled with (uppercase) letters, so that we can refer to them more easily
  - Edges that leave decision nodes (options) are employed with short descriptions of the options
- Chance nodes are left without a label or labeled with (lowercase) Greek letters
  - Edges that leave chance nodes (chances) are employed with short descriptions of chances and their probabilities
  - Sum of chance probabilities leaving a chance node is 1
- In end nodes we just write the numeric values of final outcomes

Note: In the examples of this lecture I've left everything unlabeled, since there aren't that many nodes.

Decision tree

- Example:



$$p_{11} + p_{12} = 1 \qquad p_{21} + p_{22} = 1$$

Decision trees in finance

- The most common application of this kind of decision trees is business and finance
  - What kind of investments should we do?
- When drawing these decision trees, we deal with concepts of *revenue*, *cost* and *profit*
  - Revenue (or income) = money coming in
  - Cost = money leaving out
  - Profit = revenue – cost
- In decision nodes, it is common that (at least some) options come with a cost
- Decisions may lead to chance nodes, whose outcomes (chances) generate additional (or decreased, in case of a really bad decision) revenues

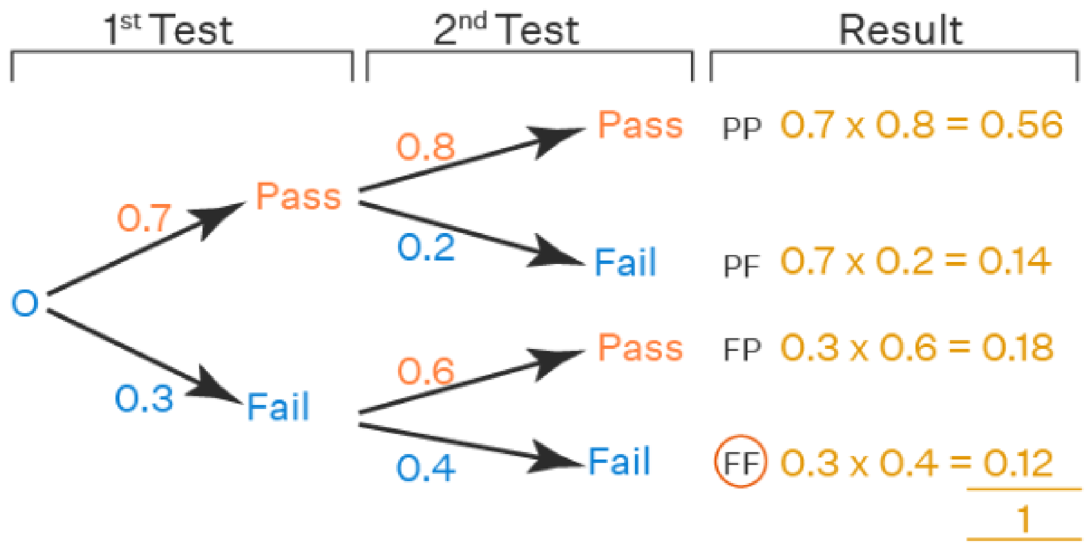
Decision trees in finance

- Two options to draw the financial decision tree
- Option 1: Profits in end nodes
  - Deduct the costs of all decisions on the decision path from final revenues in order to get profits of each path
  - Write the profits in end nodes
- Option 2: Revenues in end nodes
  - Write the revenues in end nodes
  - Write the costs of each decision in option edges (in parentheses)

- Personal opinion: **always use option 1!**
  - Less mistakes in analysis (”oops, I forgot to minus the costs”)
  - Outcomes are directly comparable

Decision tree vs. probability tree

- Decision trees resemble very much the *probability trees* we've (likely) faced in statistics and probability calculations:



- The difference is that decision tree links the probabilities to their outcomes
  - We can think that the outcome profits are weighted by their respective probabilities!

Scope notes

- Decision analysis is heavily used also in the field of artificial intelligence and machine learning
  - Classification of computer vision observations
  - Improving decision quality based on prior events
- In this decision analysis, the decision criterions are different:
  - Entropy, Gini impurity, Information gain (and others)
- In this kind of decision analysis, decisions are made repetitively, and prior decisions (and their outcomes) have an effect on the probabilities
- We'll leave this kind of evolving decision trees out from the scope of this course and concentrate on decision trees aimed for single events

Interested to read more? Check here: https://medium.com/geekculture/criterion-used-in-constructing-decision-tree-c89b7339600f

Decision rules

- The “goodness” of the decision is naturally linked to which property the decision-maker values the most
- Decision-maker evaluates the quality of decisions based on selected decision rule(s)
  - Usually only one rule is selected, but mathematically it’s possible to formulate more complicated ones, too
  - In this case we formulate an objective function that has decisions as variables and search for a minimum/maximum
- Possible decision rules:
  - Minimum risk rule
  - Rule of highest probability
  - Expected utility hypothesis

Minimum risk rule

- In *minimum risk rule*, the decision-maker goes through all options and finds out the smallest profit of each option (also known as “worst-case scenario”)
- The option that has the greatest worst-case profit (MRP) is selected
- Not very popular rule for selection, since being a pessimist and avoiding risks rarely leads to good profit
- Decent if some part of assets needs to be invested as safely as possible (in order to act as a “cushion”)
  - Risks are then taken by some other portfolio in the hope of profits

Rule of highest probability

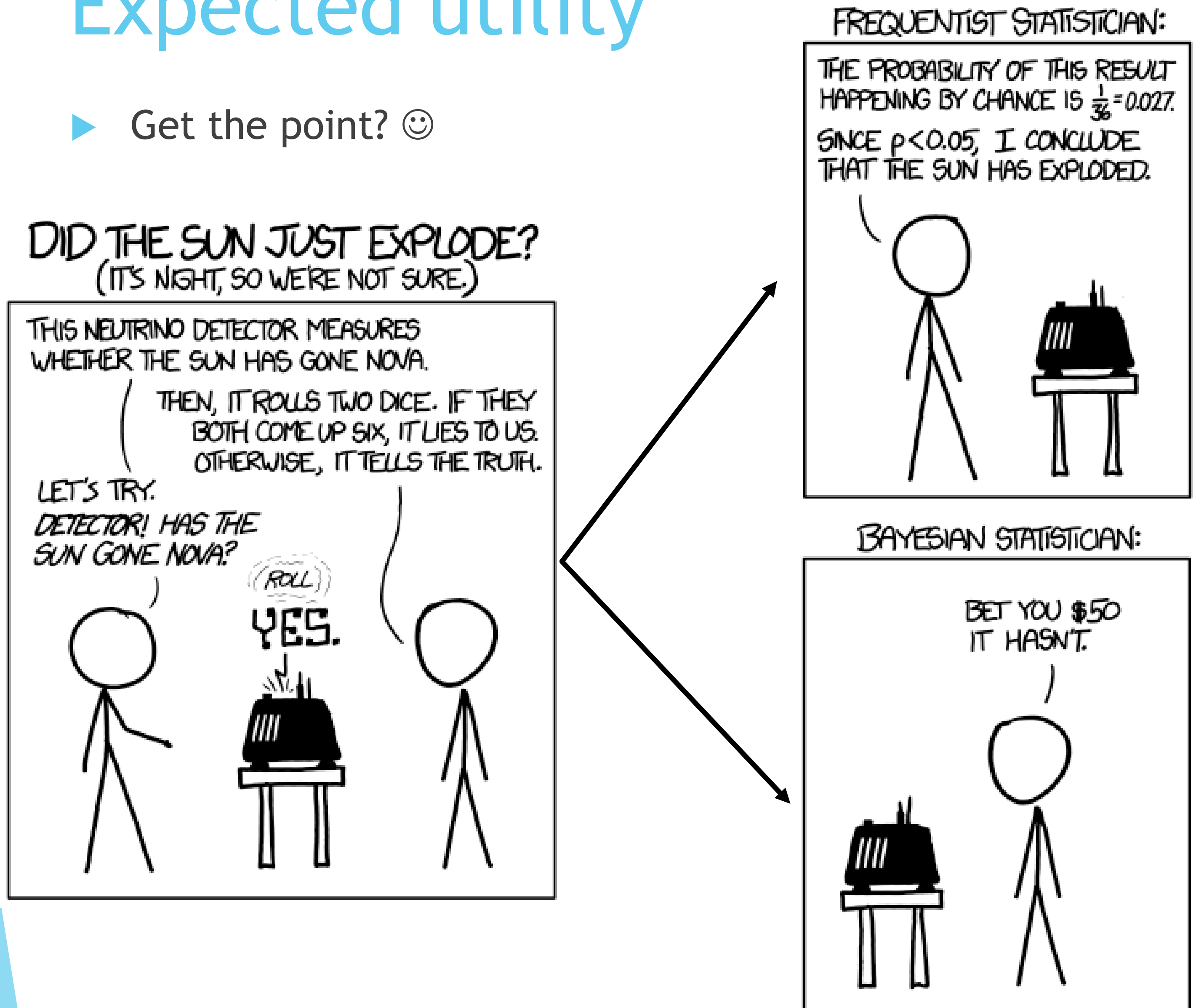
- *Rule of highest probability* starts with the assumption that for all options, the chance that has the highest probability will be the one that happens
- Best option is then decided based on this assumption: for all options, we calculate the profit in the case of chance that has the highest probability (HPP)
- The option which has the greatest HPP is selected
- One would think that this rule leads to the same results as the minimum risk rule – which it often does, but not every time

Expected utility hypothesis

- The previous rules have fundamental problems:
  - Minimum risk rule doesn't take the probabilities into account in any way
  - Rule of highest probability doesn't take into account the profit in the chance of lesser probability (even if huge)
- The rules should be combined somehow
- *Expected utility hypothesis* does exactly this:
  - Calculate the expected profit (EXP) for each option
  - Select the option which has the highest EXP
- EXP = sum of products of probability (p) & profit (P) for each chance (i) of our option
  - Mathematically: $EXP = \sum p_i P_i$
- Based on Bayesian statistics
- Best decision-making rule in the long run!

Expected utility

- Get the point? ☺

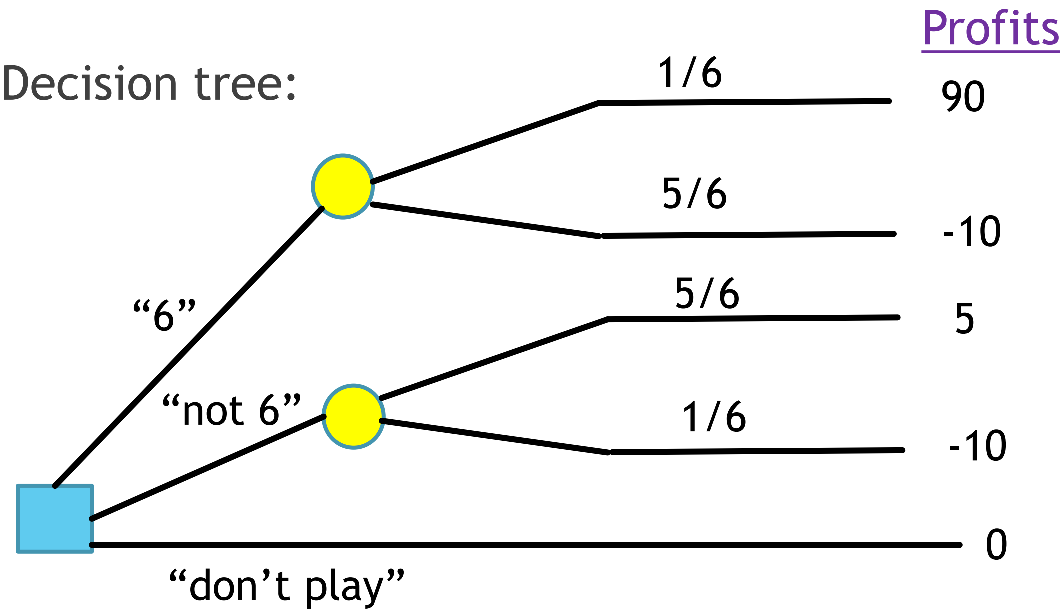


Example: Dice roll

- Casino invites each visitor to participate in a dice roll game. In the game, the participant rolls one traditional (six-face) dice once. Before the dice roll, the participant is given two options to set a $10 stake:
  - Bet on 6 – win $100
  - Bet on 1-5 – win $15
- After setting the stake, the participant rolls the dice. The stake goes to the casino anyway, but if his/her guess was correct, he/she collects the win.
- Draw a decision tree and explain, which option is taken by a visitor, who makes decisions based on
  a) Minimum risk rule
  b) Rule of highest probability
  c) Expected utility hypothesis

Example: Dice roll

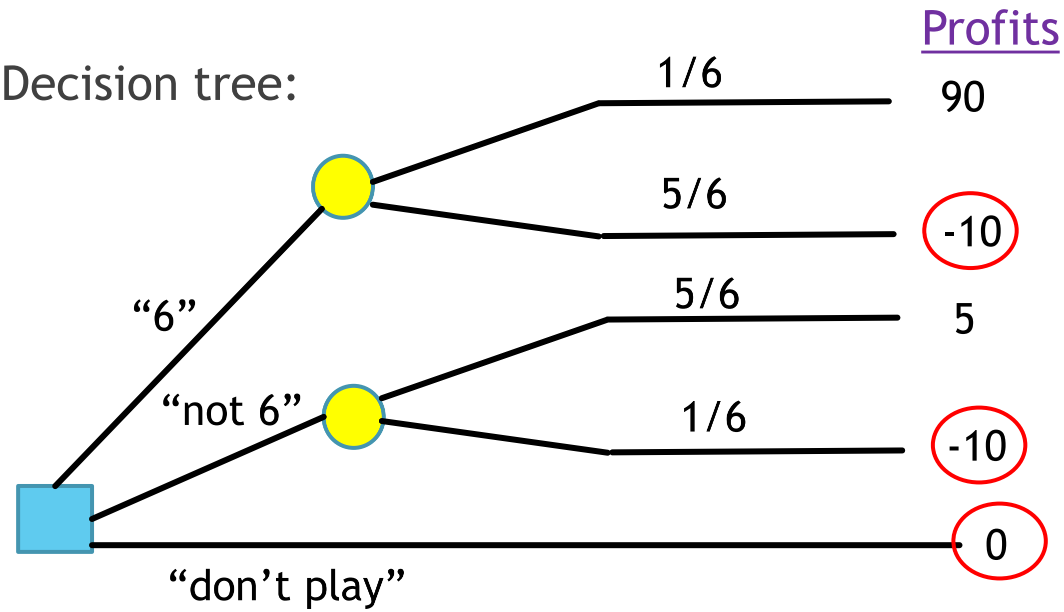
- Decision tree:



Let's examine which option should we choose when using each rule:

Example: Dice roll

- Decision tree:

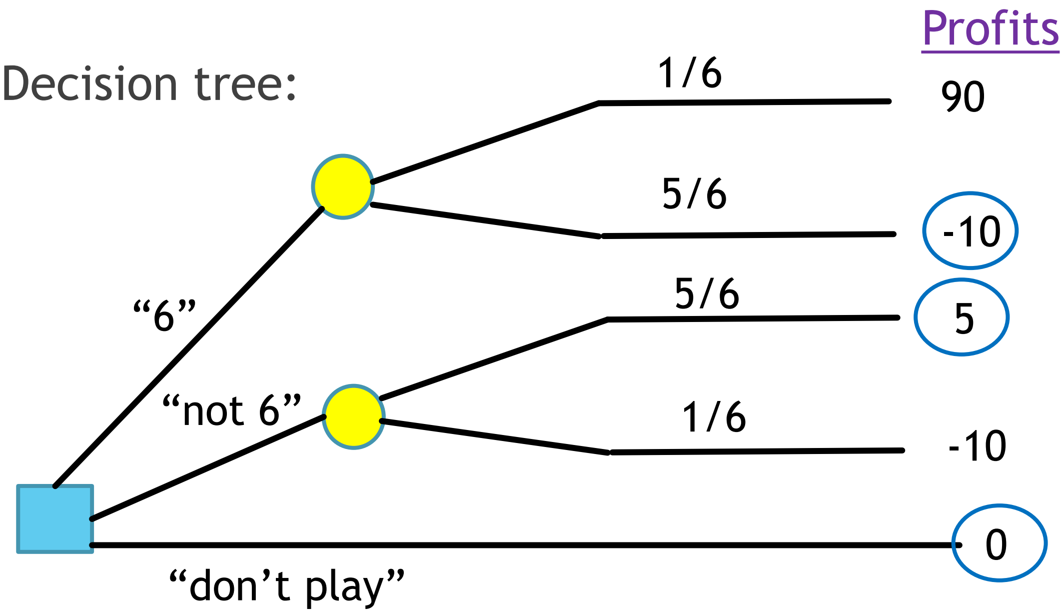


a) Minimum risk rule:
MRP(“6”) = -10
MRP(“not 6”) = -10
MRP(“don’t play”) = 0 (greatest!)

→ decision: don’t participate in the game

Example: Dice roll

- Decision tree:

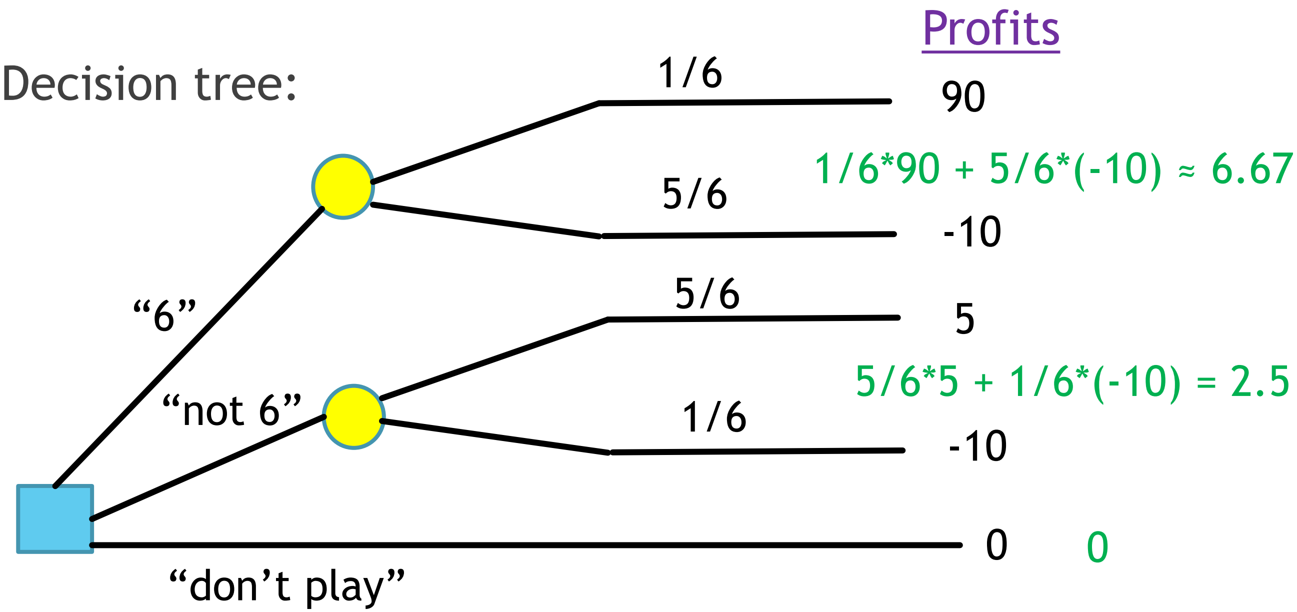


b) Rule of highest probability:
HPP("6") = -10
HPP("not 6") = 5 (greatest!)
HPP("don't play") = 0

→ decision: participate, bet on "not 6"

Example: Dice roll

- Decision tree:



c) Expected utility hypothesis:
EXP("6") = 6.67 (greatest!)
EXP("not 6") = 2.5
EXP("don't play") = 0

→ decision: participate, bet on "6"

Example: Search for oil 1

- Company Inc. owns a land property which it doesn't need. Anyhow, according to reports given to the board of directors, there's a 25 % probability that there is a small oil well on the land. This is a tempting opportunity, because according to estimates, then the company would get $800k of income for selling the oil. The downside is that organizing drilling works would cost $100k, and if no oil is found, this money is wasted.
- A 3rd party investor has offered to buy the land for $90k. The company has a hunch that this interest is due to the suspected oil well.
- The board of directors gathers in a meeting and tries to decide what to do in the situation.

Example: Search for oil 1

- The board of directors decides to draw a decision tree:
  - If they drill and oil is found, profit is 800 – 100 = 700
  - If they drill and no oil is found, the drilling costs are lost, so the profit is -100. Added to that, presumably also the interest of the 3rd party investor is lost (= can't sell the land anymore)
  - If they sell, the profit is 90 (sell price)

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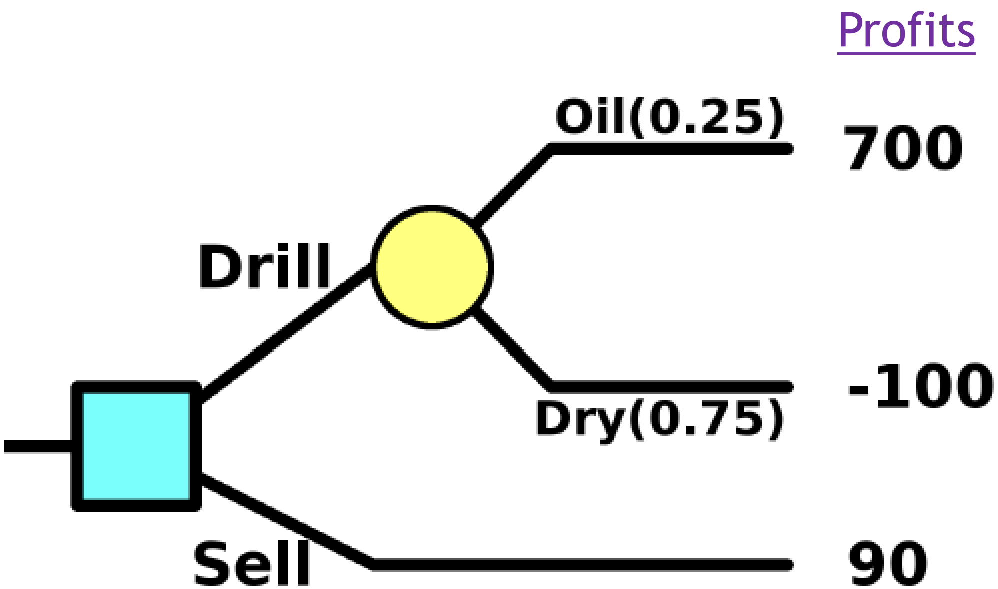

Example: Search for oil 1

- Let's examine the decision by using multiple decision rules:
  - Minimum risk rule would lead to selling ($90k sure profit)
  - Rule of highest probability would assume that no oil is found (75 % vs. 25 %), so drilling would only result in loss of $100k. Therefore, according to this rule we should also sell the land.
  - Expected utility hypothesis requires calculation of EXPs:

Example: Search for oil 1

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$$EXP(Drill) = 0.25 \cdot 700 + 0.75 \cdot (-100) = 175 - 75 = 100$$

$$EXP(Sell) = 90$$

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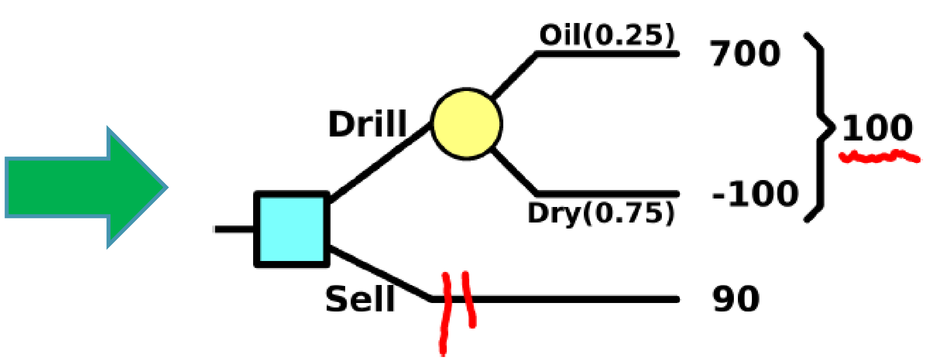

Decision: Drill and see what happens!

Stability analysis

- Probabilities of chances in the decision tree are seldom surely known values, but are based on estimates
- If the probabilities change even a little bit, it is possible that the optimal decision changes
- Therefore, it is important to do a *stability analysis* for the decision tree:
  - Denote the probability of our chance by p
  - Plot the EXPs as functions of p
  - Which option produces the greatest EXP by which values of p?

Stability analysis

- For example, the previous search for oil problem: denote the probability of oil well by p
  - Expected profit of drill option is

$$EXP(Drill) = p \cdot 700 + (1 - p) \cdot (-100)$$

$$= 700p + 100p - 100 = \mathbf{800p - 100}$$

  - Expected profit of sell option is 90 (constant)

- When are the EXPs equal?

Stability analysis

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$$EXP(Drill) = p \cdot 700 + (1-p) \cdot (-100)$$

$$= 700p + 100p - 100 = \mathbf{800p - 100}$$

  - Expected profit of sell option is 90 (constant)

- When are the EXPs equal?

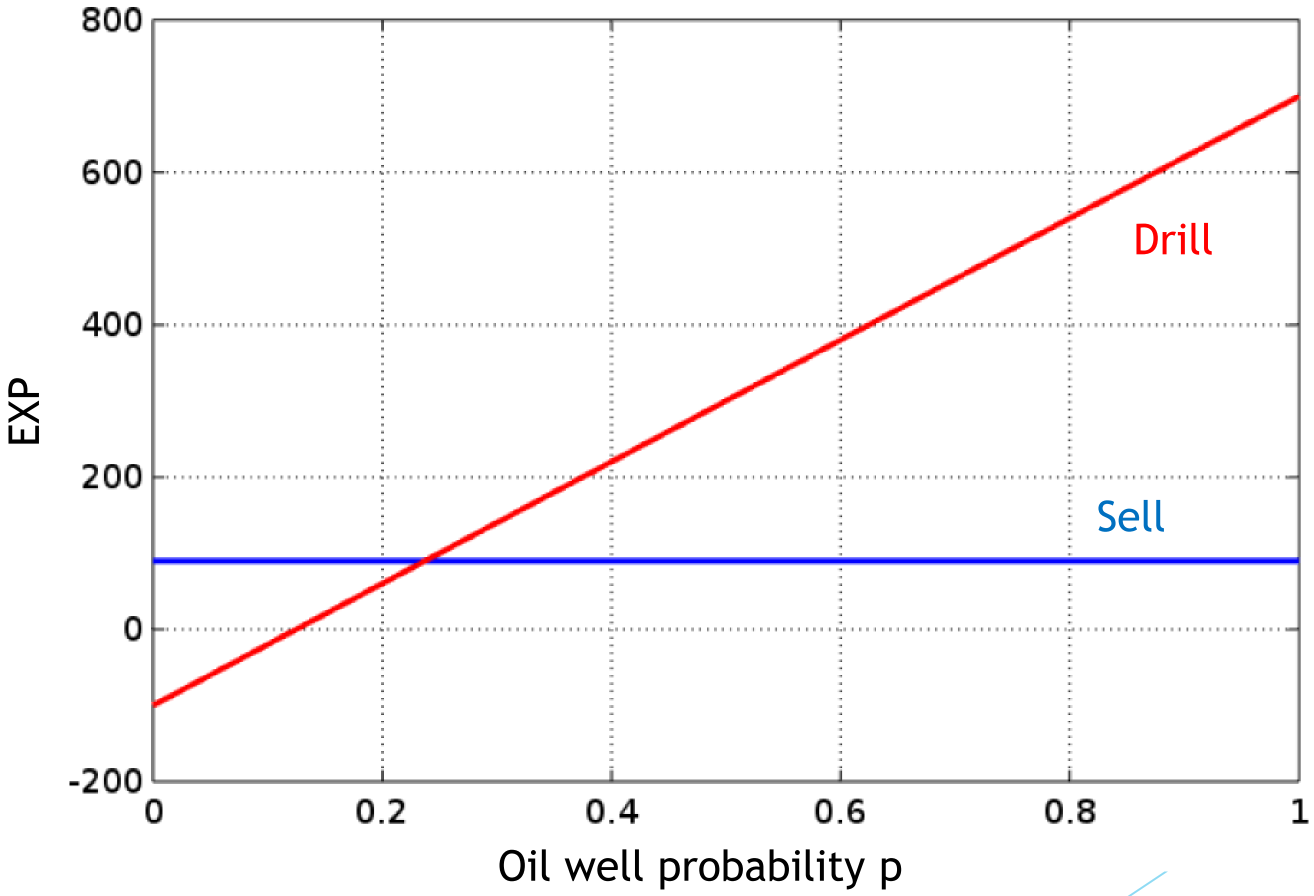
$$EXP(Drill) = EXP(Sell)$$

$$800p - 100 = 90$$

$$p = \frac{190}{800} = 0.2375$$

Stability analysis

- Both EXPs plotted in the same picture:



Several decision nodes

- It is very common that our decision tree is not as simple as in prior examples; usually decision trees contain several decision nodes (and several chance nodes)
- In this kind of cases, we start analyzing the decision tree from the right:
  - Calculate EXPs for last choices and select the best of them (cut off worse ones)
  - Using these EXPs, calculate the EXPs for preceding choices and select the best of them (cut off worse ones)
  - Continue until we end up in the root; at this point we have found out the best choices for each decision node

Example: Search for oil 2

- Let's expand the previous example a bit by taking into account one more option: getting more information by performing additional analyses
- Geotechnical expert company SeisCo offers to perform a seismic survey for the land property for $30k. This survey is not going to give us a bulletproof answer on the existence of the oil well, but it provides an estimate on whether the soil is favorable for oil wells. SeisCo presents the following number claims:
  - If there is oil in the land, the survey gives a favorable result with a probability of 60 %
  - If there is no oil in the land, the survey gives an unfavorable result with a probability of 80 %

Example: Search for oil 2

- Let's use abbreviations FSS = "Favorable seismic survey" and USS = "unfavorable seismic survey"
- Then the probabilities given by SeisCo are

$$P(USS \mid Oil) = 0.4$$
$$P(FSS \mid Oil) = 0.6$$
$$P(USS \mid Dry) = 0.8$$
$$P(FSS \mid Dry) = 0.2$$

- Considering the decision tree, these probabilities are the wrong way around; we'd want to know the probabilities in the form "if the survey gives FSS, then there is oil in the land with a probability of …"

Example: Search for oil 2

- So, we want to convert the conditional probabilities:

$$P(FSS \mid Oil) \Rightarrow P(Oil \mid FSS)$$

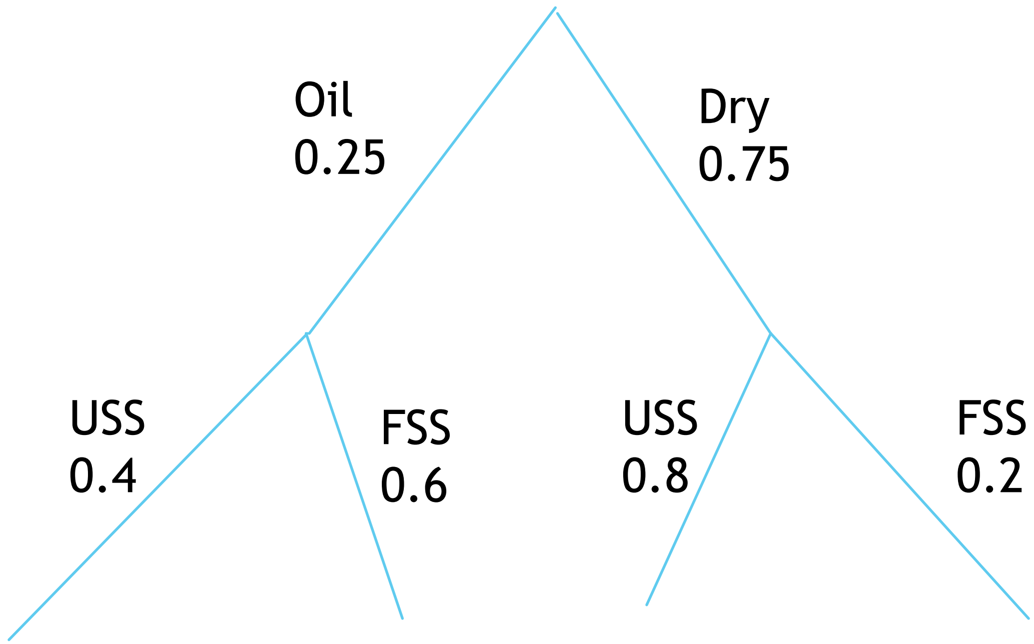
- Luckily this can be done very easily using rules of statistics – namely, using the Bayes theorem:

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)} = \frac{p(B|A)\ p(A)}{p(B)}$$

- In order to solve this, we only need to find out the total probability of B
  - Can be solved f. ex. by using a probability tree

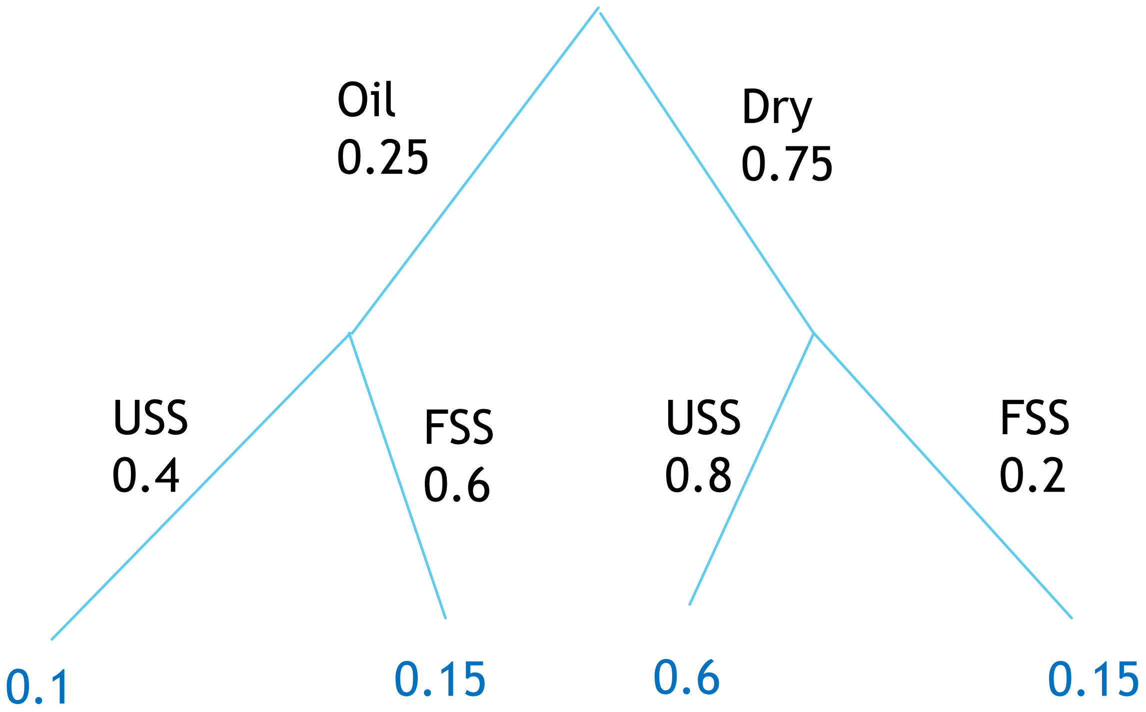
Example: Search for oil 2

- Draw the probability tree:



Example: Search for oil 2

- Draw the probability tree:



P(USS) = 0.1 + 0.6 = 0.7
P(FSS) = 0.15 + 0.15 = 0.3

Example: Search for oil 2

- Now we can calculate the probabilities of different choices to our decision tree:

$$P(Oil\ |USS) = \frac{P(Oil \cap USS)}{P(USS)} = \frac{0.1}{0.7} \approx 0.143$$

$$P(Dry\ |USS) = \frac{P(Dry\ \cap USS)}{P(USS)} = \frac{0.6}{0.7} \approx 0.857$$

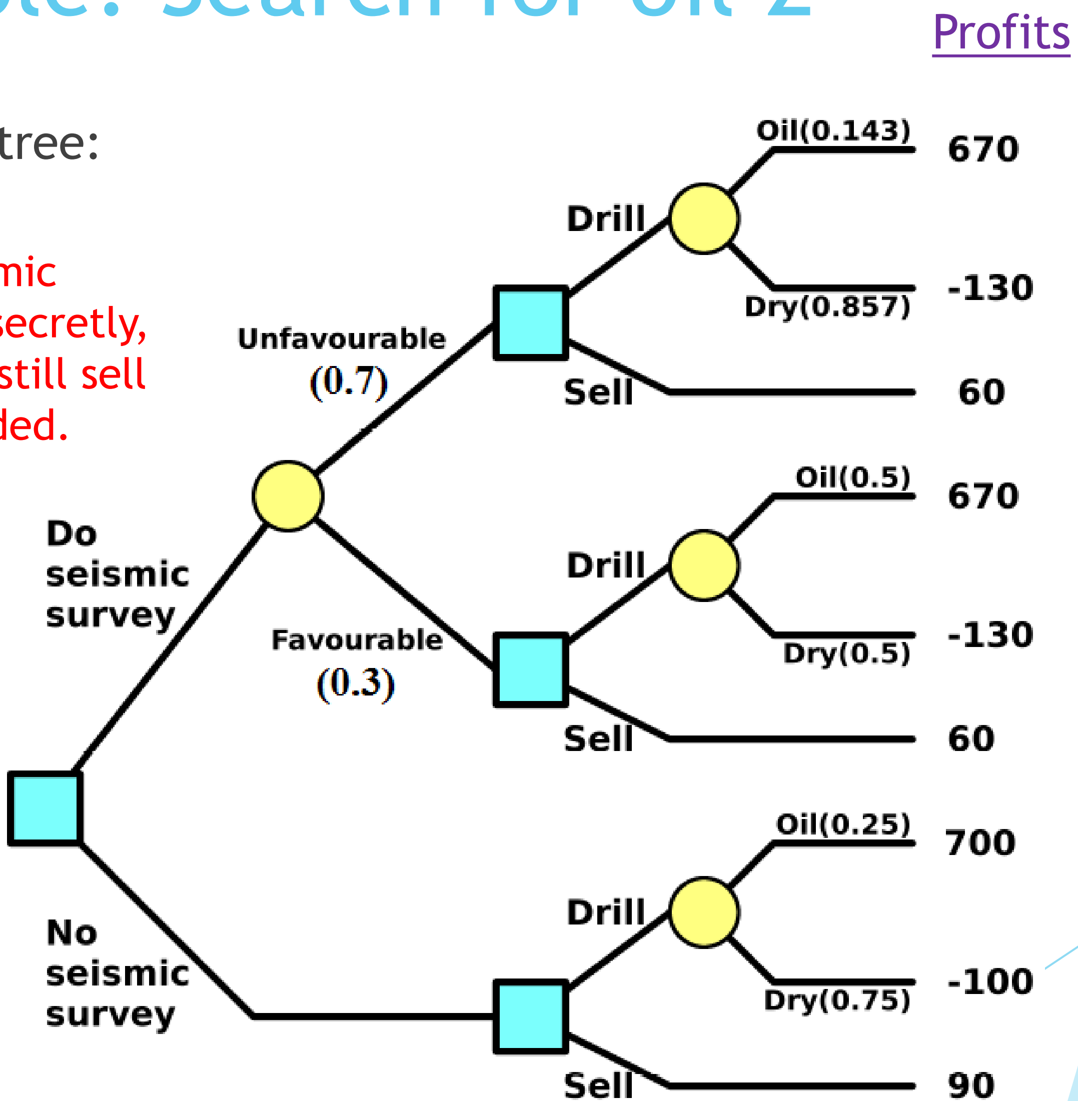
$$P(Oil\ |FSS) = \frac{P(Oil \cap FSS)}{P(FSS)} = \frac{0.15}{0.3} = 0.5$$

$$P(Dry\ |FSS) = \frac{P(Dry \cap FSS)}{P(FSS)} = \frac{0.15}{0.3} = 0.5$$

Example: Search for oil 2

- Decision tree:

NOTE: The seismic survey is done secretly, so that we can still sell the land if needed.



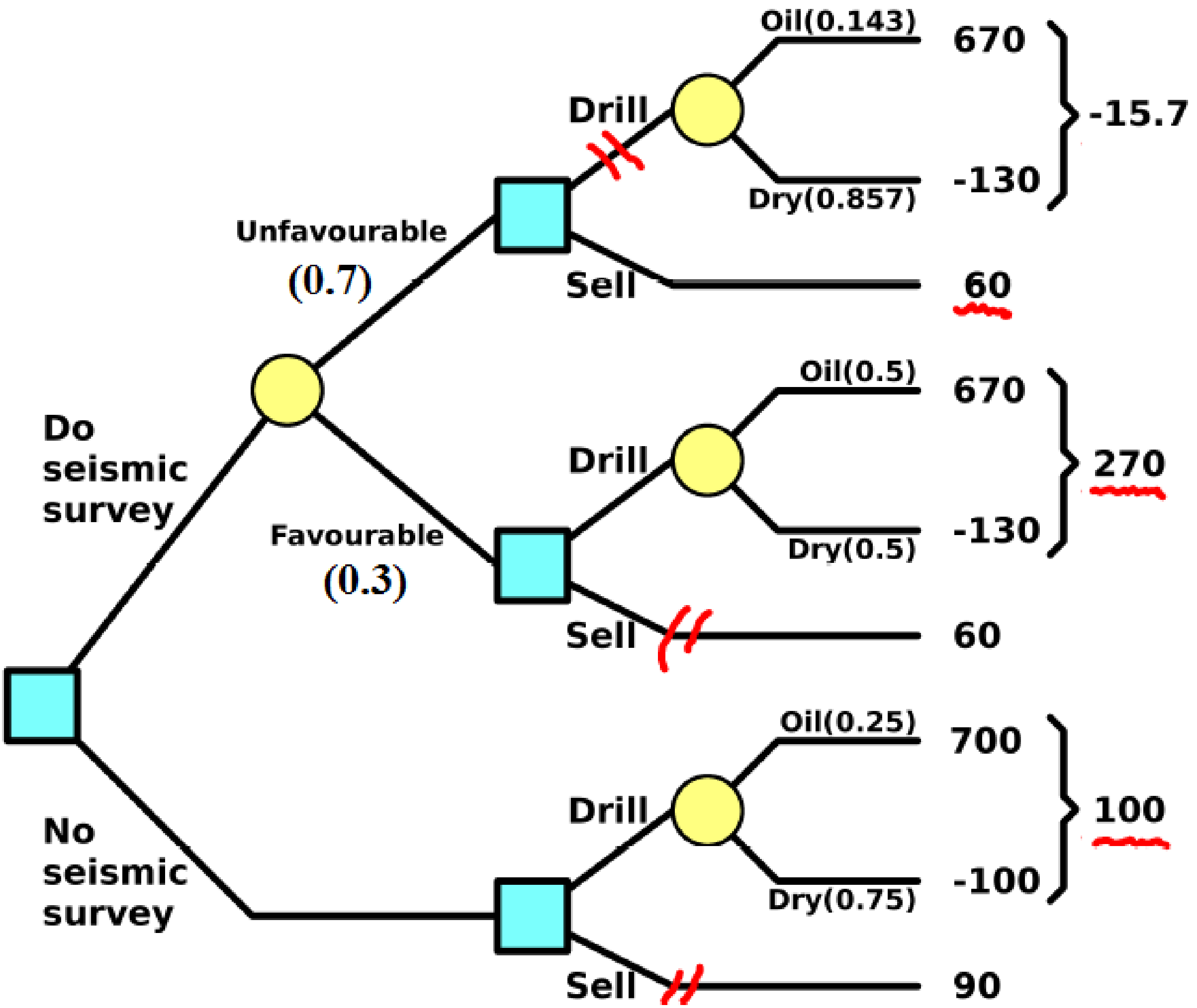
Example: Search for oil 2

- Calculate the EXPs of rightmost decision node choices:
  - EXPs only need to be calculated for options which end up in chance nodes; therefore only “Drill” options are calculated here (EXPs of “Sell” options are trivial)

$$EXP(SS, U, Drill) = 0.143 \cdot 670 + 0.857 \cdot (-130) = -15.7$$
$$EXP(SS, F, Drill) = 0.5 \cdot 670 + 0.5 \cdot (-130) = 270$$
$$EXP(No\ SS, Drill) = 0.25 \cdot 700 + 0.75 \cdot (-100) = 100$$

- Write this information to the decision tree and cut off the worse options
- After this, write the EXPs of the best options to next decision nodes on the left

Example: Search for oil 2

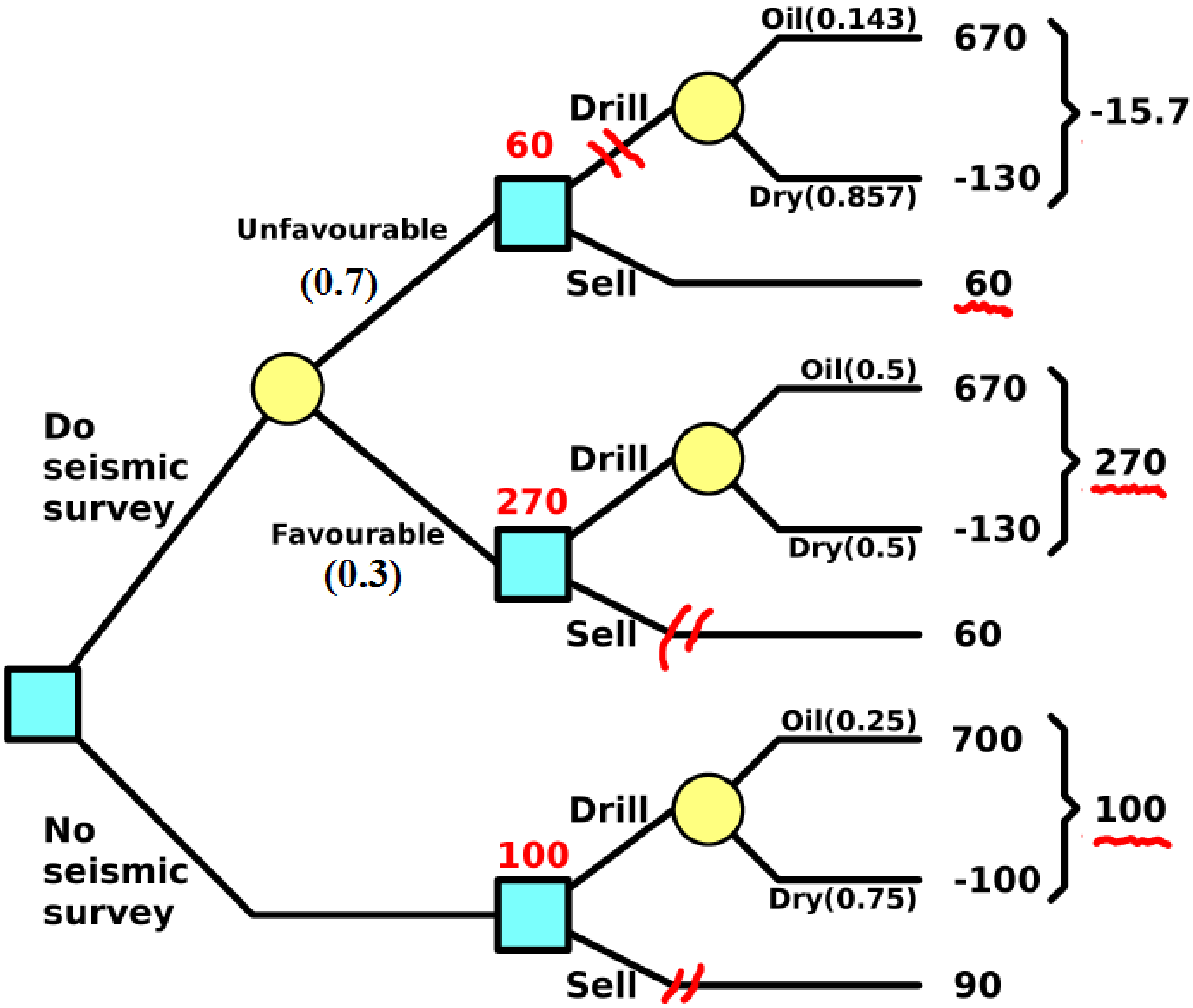


$60 \succ -15.7$

$270 \succ 60$

$100 \succ 90$

Example: Search for oil 2



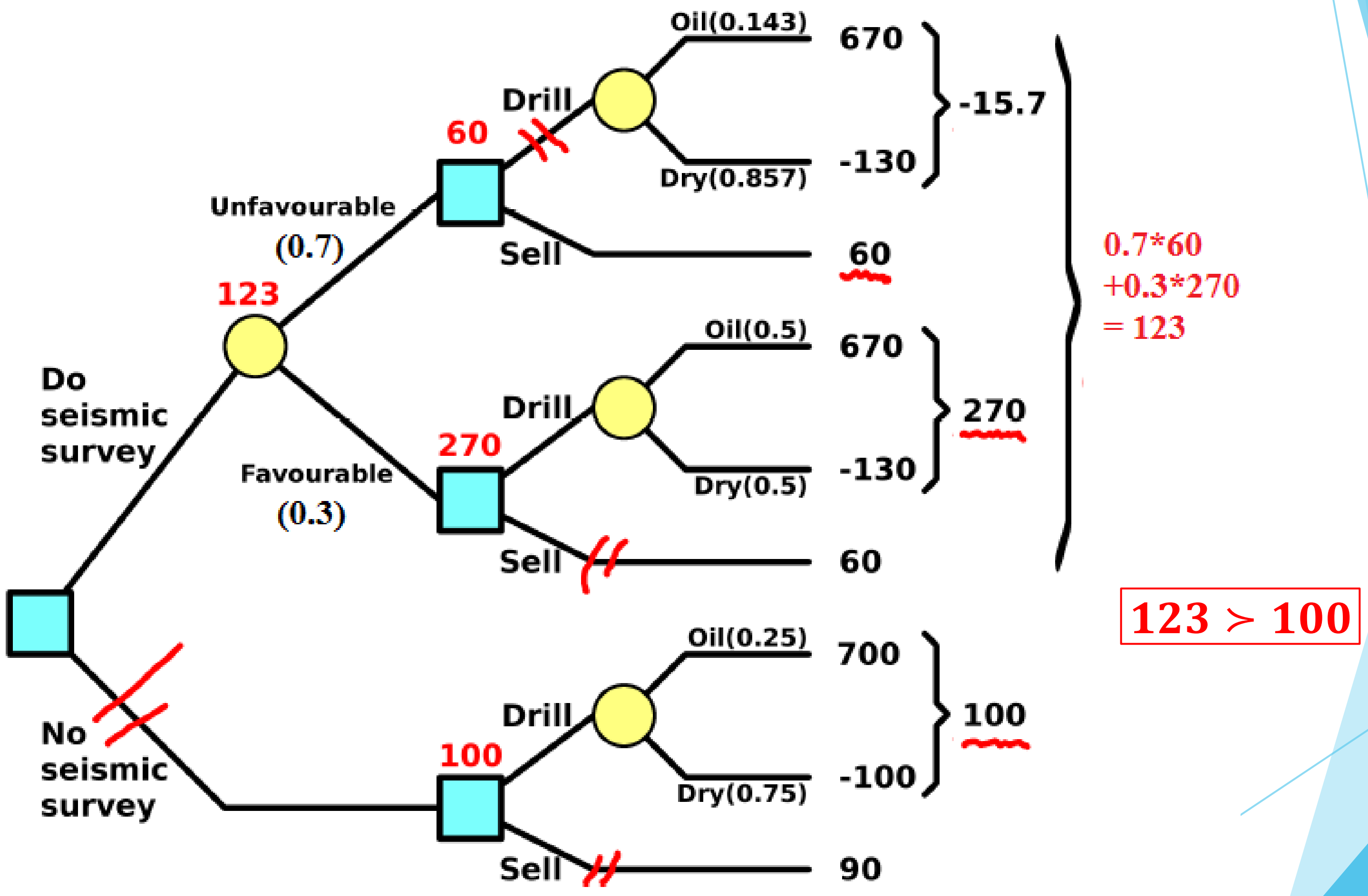
Example: Search for oil 2

- Then continue our journey towards the root and calculate the EXP for seismic survey:

$$EXP(SS) = 07 \cdot 60 + 0.3 \cdot 270 = 123$$

- Mark this to the decision tree and compare the options “seismic survey” and “no seismic survey”
- Again, cut the worse branch(es) away
- Now our analysis is almost finished! What we still need to report are the conclusions:

Example: Search for oil 2



Example: Search for oil 2

- Conclusion: the best plan is to
  - Do the seismic survey (EXP 123 > 100)
  - If the result of seismic survey is favorable, drill (270 > 60)
  - If the result of seismic survey is unfavorable, we keep our mouths shut and sell the land to 3rd party investor (60 > -15.7)

- NOTE: The final answer must be a full roadmap on what to do in which case! (Because we can't control the chance outcomes)
- NOTE 2: This doesn't guarantee that the profit will be optimal – the result of seismic survey could have been a false positive, and the new owner can then drill and gather the $800k (minus purchase price and drilling costs, naturally)
  - Statistically it is next to impossible to be always right
  - Anyway, in the long run, this method will produce the best profits on average

Thank you!