Foundations of Information Processing

Propositional logic and reasoning

Considerations about logic and reasoning

Consideration 1:

How to change
expressions of natural languages
to formal expressions?

Consideration 2:

What is
propositional logic?

How is it related to reasoning?

Consideration 3:

How to infer mathematically?

Can we automate reasoning?

Logic studies rules of written or formal claims and conclusions.

To be able to apply mathematical logic, rather free-form expressions of a natural language have to be changed to exact formal expressions.

Methods of reasoning are different and properties of logic limit applying the methods in practice.

Logic and reasoning

- Modeling logical reasoning.
  - Could we automate thinking?
- Propositional logic.
  - The tool of modelling for automating logical thinking.
- Propositions and connectives.
  - Logical sentences (statements) from the premises.
  - Combined by connectives like operators in mathematics.
- Truth tables: values of propositions (truth values).
- Simplification rules: changing propositions to be more useful.
- Rules of inference: what can be inferred from what?
- Logical reasoning: is the conclusion true?
  - Semantic method.
  - Syntactic method.
  - Resolution method.

About the history of logic (of quite a long one)

- The term “logic” is from a Greek word λογική (logikē) which is derived from the word λόγος (logos) (“word”, “order”, “reason”).
- The logic was introduced in China, India, and Greece in 400-100 BC.
- Philosophical logic and mathematical logic.
- Prof. Gottfried Wilhelm Leibniz (1646-1716), a German mathematician and philosopher, the development of universal philosophical thinking.
- Boolean logic:
  - Prof. George Boole, a British mathematician and philosopher, 1815-1864.
  - Lincoln Mechanics' Institution, England ja Queen's College, Cork, Ireland.
  - One of the founders of computer science: the Boolean algebra is the foundation of computer arithmetic.
  - The Laws of Thought (1854).
  - Tools to formalize logical thinking.
  - Boole got wet in rain on his way to his university and died in pneumonia.

Propositional logic

- Motivation: could facts and rules be modeled logically to make conclusions (in selected applications)?
- Propositional logic (also called as propositional calculus, statement logic, and zeroth-order logic) is a formal language.
- Statements that contain one proposition only are called atomic propositions.
- Atomic propositions can be combined by connectives, generating new propositions.
- The inference is a process where from one or more premises are reasoned one or more conclusions.
- Rules of inference unambiguously define
  - what kind of a conclusion can be reasoned from the premises, and
  - whether the conclusion is true or false.

Notations of logic

- Atomic propositions are defined by symbols.
  - For example, p, q, ...
  - The propositions contain logical content.
  - For example, p = ”it is raining” ja q = ”it is windy”.
- Each atom proposition contains a logical value:
  - true (T, 1) or
  - false (F, 0).
- New propositions can be built from atom propositions (premises) using logical connectives.
  - For example: $p \wedge q$
- Assuming that the premises are true, applying the rules of inference it can be proven whether the conclusion is true or false.

Propositions

	T or F?
The Earth is flat and the Moon orbits Mars	
The Earth is round and the Moon orbits Mars	
The Earth is flat and the Moon orbits the Earth	
The Earth is round and the Moon orbits the Earth	
The Earth is flat and the Moon orbits the Earth	
The Earth is round and the Moon does not orbit Mars	
The Earth is not flat or the Moon does not orbit the Earth	
The Moon orbits the Earth and the Earth orbits the Sun	

The truth table: all combinations of truth values with a chosen connective

x	y	AND
The Earth is not round	The Moon does not orbit the Earth	F
The Earth is not round	The Moon orbits the Earth	F
The Earth is round	The Moon does not orbit the Earth	F
The Earth is round	The Moon orbits the Earth	T

Logical connectives

Negation	¬	$\neg P$	Not *P*	NOT
Conjunction	∧	$P \wedge Q$	*P* and *Q*	AND
Disjunction	∨	$P \vee Q$	*P* or *Q*	OR
Conditional	→	$P \rightarrow Q$	if *P* then *Q*	
Biconditional	↔	$P \leftrightarrow Q$	*P*, if and only if *Q*	
Peirce's arrow	↓	$P \downarrow Q$	not (*P* or *Q*)	NOR
Sheffer stroke	↑	$P \uparrow Q$	not (*P* and *Q*)	NAND
Exclusive disjunction	⊕ ($\bar{\vee}$)	$P \oplus Q$	either *P* or *Q*	XOR

Truth tables: negation (NOT), conjunction (AND), disjunction (OR)

not *p*

p	$\neg p$
F	T
T	F

p and *q*

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

p or *q*

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Conditional →

If p then q

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

$p \rightarrow q$

"the principle of a broken promise":

false only when true => false

Biconditional ↔

p, if and only if q

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$ $(p \rightarrow q) \wedge (q \rightarrow p)$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

Peirce's arrow ↓ (Not OR, NOR, $\bar{\vee}$)

- Not *(p* or *q)*:

p	q	$p \vee q$	$\neg(p \vee q)$
F	F	F	T
F	T	T	F
T	F	T	F
T	T	T	F

Truth tables:
conditional, biconditional, Peirce's arrow (NOR)

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

p	q	$p \downarrow q$
F	F	T
F	T	F
T	F	F
T	T	F

Sheffer stroke ↑ (Not AND, NAND)

- Not (*p* and *q*):

p	q	$p \wedge q$	$\neg(p \wedge q)$
F	F	F	T
F	T	F	T
T	F	F	T
T	T	T	F

Truth tables:
Scheffer stroke (NAND) and Exclusive OR (XOR)

p	q	$p \uparrow q$
F	F	T
F	T	T
T	F	T
T	T	F

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

Connectives: order of precedence

- The order of precedence is as follows (¬ first and ↔ last):

$$\neg$$

$$\wedge$$

$$\vee$$

$$\rightarrow$$

$$\leftrightarrow$$

- The parentheses () are considered first.
  - $a \wedge (b \vee c)$ where $b \vee c$ is calculated first.
  - $a \wedge b \vee c$ where $a \wedge b$ is calculated first.

Methods of logical reasoning (inference)

- Semantic method (cf. meaning):
  - The truth table.
  - Combinations of all possible truth values of atomic propositions used.
- Syntactic method (cf. grammar):
  - Two options:
    - Simplification of propositions.
    - Rules of inference.
- Resolution method:
  - One rule of inference only used (resolution rule).
  - To be considered in more details in the course "Foundations of Computer Science".

How to prove the conclusion?

- If the proposition is true in all *cases* (whatever the combination of the truth values of the atom propositions are)

  => the proposition is *tautology*.
- If the proposition is false in all cases

  => the proposition is *contradiction*.
- Otherwise, the proposition is *contingent*.
- If **the proposition** is **tautology**

  => **the conclusion** proposed by this proposition is **true**.
- Tautology can be proven as follows:
  - Using the truth table (the semantic method).
  - Simplifying the proposition to become true (the syntactic method).

TAUTOLOGY				
p	q	p∧q	p∨q	(p∧q)→(p∨q)
F	F	F	F	T
F	T	F	T	T
T	F	F	T	T
T	T	T	T	T

CONTINGENT				
P	q	p∧q	p→q	(p∧q)∧(p→q)
F	F	F	T	F
F	T	F	T	F
T	F	F	F	F
T	T	T	T	T

CONTRADICTION						
p	q	p∨q	¬(p∨q)	p→q	¬(p→q)	¬(p∨q)∧¬(p→q)
F	F	F	T	T	F	F
F	T	T	F	T	F	F
T	F	T	F	F	T	F
T	T	T	F	T	F	F

Example: the semantic method

- Prove using the truth table that r can be concluded from the following premises:

$$p \wedge q \rightarrow r$$
$$q \rightarrow p$$
$$q$$

- Thus, from the premises $p \wedge q \rightarrow r$ ja $q \rightarrow p$ and q it can be inferred the conclusion r

  $\Rightarrow$ prove that the following is tautology:

$$((p \wedge q \rightarrow r) \wedge (q \rightarrow p) \wedge q) \rightarrow r$$

- This can be proven using *the semantic method* as follows:
  - Is the proposition true with all possible combinations of the atomic propositions?
  - If yes, then it is tautology.

Truth table: $((p \wedge q \rightarrow r) \wedge (q \rightarrow p) \wedge q) \rightarrow r$

p	q	r	$p \wedge q$	$p \wedge q \rightarrow r$	$q \rightarrow p$	$(p \wedge q \rightarrow r) \wedge (q \rightarrow p)$	$(p \wedge q \rightarrow r) \wedge (q \rightarrow p) \wedge q$	$(p \wedge q \rightarrow r) \wedge (q \rightarrow p) \wedge q \rightarrow r$
F	F	F	F	T	T	T	F	**T**
F	F	T	F	T	T	T	F	**T**
F	T	F	F	T	F	F	F	**T**
F	T	T	F	T	F	F	F	**T**
T	F	F	F	T	T	T	F	**T**
T	F	T	F	T	T	T	F	**T**
T	T	F	T	F	T	F	F	**T**
T	T	T	T	T	T	T	T	**T**

This method can be automated.

Is this way of reasoning feasible?

Does it work in practice?

What is the challenge?

Complexity of the sematic method

- The fundamental question:
  - How many combinations must be checked when there are *n* atomic propositions?
- Each atomic proposition can have two possible truth values (true/false) so the total number of combinations is as follows:

$$2^n$$

- The computational complexity is thus *exponential*.
- This means that the method is not computable in practice.
- In our example, $2^3 = 8$, but when $n = 20$ there are more than one million options to be checked.
- Would there be some other way to reason logically? Something based on other logical rules?

Simplification rules of propositions

$$1)\ p \wedge p \equiv p \quad p \vee p \equiv p$$
$$2)\ p \wedge q \equiv q \wedge p \quad p \vee q \equiv q \vee p$$
$$3)\ (p \wedge q) \wedge r \equiv p \wedge (q \wedge r) \quad (p \vee q) \vee r \equiv p \vee (q \vee r)$$
$$4)\ p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \quad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$5)\ p \wedge (p \vee q) \equiv p \quad p \vee (p \wedge q) \equiv p$$
$$6)\ p \wedge (\neg p \vee q) \equiv p \wedge q \quad p \vee (\neg p \wedge q) \equiv p \vee q$$
$$7)\ \neg(p \wedge q) \equiv \neg p \vee \neg q \quad \neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$8)\ \neg(\neg p) \equiv p$$
$$9)\ p \rightarrow q \equiv \neg p \vee q$$
$$10)\ p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$11)\ p \wedge \neg p \equiv E \quad p \vee \neg p \equiv T$$
$$12)\ p \wedge T \equiv p \quad p \vee E \equiv p$$
$$13)\ p \wedge E \equiv E \quad p \vee T \equiv T$$

Example: the syntactic method using simplification of propositions

- Prove using simplification of propositions that r can be concluded from the following premises:

$$p \wedge q \to r$$
$$q \to p$$
$$q$$

- The simplification rules available are numbered on the previous page.
- On the next page, the used simplification rules to prove the conclusion are numbered accordingly.

Simplification

$$\big(((p \wedge q) \rightarrow r) \wedge (q \rightarrow p) \wedge q\big) \rightarrow r \qquad (9)$$

$$\big((\neg(p \wedge q) \vee r) \wedge (\neg q \vee p) \wedge q\big) \rightarrow r \qquad (7)$$

$$\big((\neg p \vee \neg q \vee r) \wedge (\neg q \vee p) \wedge q\big) \rightarrow r \qquad (9)$$

$$\neg((\neg p \vee \neg q \vee r) \wedge (\neg q \vee p) \wedge q) \vee r \qquad (7)$$

$$(\neg(\neg p \vee \neg q \vee r) \vee \neg(\neg q \vee p) \vee \neg q) \vee r \quad (7{,}8)$$

$$(p \wedge q \wedge \neg r) \vee (q \wedge \neg p) \vee \neg q \vee r \qquad (4)$$

$$q \wedge ((p \wedge \neg r) \vee \neg p) \vee \neg q \vee r \qquad (6)$$

$$q \wedge (\neg r \vee \neg p) \vee \neg q \vee r \qquad (4)$$

$$(q \wedge \neg r) \vee (q \wedge \neg p) \vee \neg q \vee r \qquad (6)$$

$$(q \vee r) \vee (\neg p \vee \neg q) \qquad \text{remove ()}$$

$$q \vee r \vee \neg p \vee \neg q \qquad (11)$$

$$T \vee (r \vee \neg p) \qquad (13)$$

$$T$$

Quite a job to do!

Example: the syntactic method using rules of inference

- Logical reasoning seemed to be quite laborious using the simplification of propositions.
  - Could there be another way?

  $\Rightarrow$ logical rules of inference
- Premises $\Sigma = \{P_1, P_2, \ldots, P_n\}$.
- Conclusion Q.
- The conclusion is true when the premises are proven to be true, leading to the conclusion:

  $$P_1, P_2, \ldots, P_n \Rightarrow Q$$
- The rules of inference are needed to go from the premises to the conclusion. See the next page.

Syntactic method: rules of inference

Premise or already concluded	Conclusion $\Rightarrow$	Abbreviation	The name of the rule
P, Q	$P \wedge Q$	CI	Conjunction's introduction
$P \wedge Q$	P, Q	CE	Conjunction's elimination
P	$P \vee Q$	DI	Disjunction's introduction
$P \vee Q$ ja $P \rightarrow R$ ja $Q \rightarrow R$	R	DE	Disjunction's elimination
Q is concluded from P	$P \rightarrow Q$	II	Implication's introduction
P, $P \rightarrow Q$	Q	MP	Modus Ponens
P is concluded to be false	$\neg P$	NI	Negation's introduction
P, $\neg P$	false	NE	Negation's elimination
$P \rightarrow Q$, $\neg Q$	$\neg P$	MT	Modus Tollens

Syntactic method: How to use the rules?

- Reason syntactically r from the following premises:

$$p \wedge q \rightarrow r \qquad q \rightarrow p \qquad q$$

- Number the premises and apply the rules of inference step by step:

1. Premise: $p \wedge q \rightarrow r$
2. Premise: $q \rightarrow p$
3. Premise: q
4. Steps 3 and 2; Modus Ponens: $\begin{matrix} P \\ P \rightarrow Q \end{matrix} \Rightarrow Q \qquad \begin{matrix} q \\ q \rightarrow p \end{matrix} \Rightarrow p$
5. Steps 4 and 3; Conjunction's Introduction: $\begin{matrix} P \\ Q \end{matrix} \Rightarrow P \wedge Q \qquad \begin{matrix} p \\ q \end{matrix} \Rightarrow p \wedge q$
6. Steps 5 ja 1; Modus Ponens: $\begin{matrix} P \\ P \rightarrow Q \end{matrix} \Rightarrow Q \qquad \begin{matrix} p \wedge q \\ p \wedge q \rightarrow r \end{matrix} \Rightarrow r$

Further steps of reasoning

- The semantic method is exponential.

  => not computational in practice.
- It is challenging to automate (to mechanize) the syntactic method.

  => what would be the sequence of the rules to be used?
- What to do next?

  => the resolution method.
- The resolution method uses one rule only

  $$(A \to B) \land (B \to C) \Rightarrow (A \to C)$$

  and applies the proof by contradiction.
- The resolution method is considered in the course "Foundations of Computer Science".
- Moreover, better definitions to logical statements are needed.
  - Why? See the next page.

Extending definitions of logical statements

- Better definitions are needed to logical statements.
- How can be defined whether a person *x* is a man: *man(x)*?
  - The variable inside the proposition must be introduced:

    man(Heikki) is true, *man(Katarina)* is false.
  - This is called predicate logic (or first-order logic): more details in the course “Foundations of Computer Science.”
- Is everything exactly true or false?
  - Is a person tall?
  - Flexible truth values are needed => fuzzy logic!
  - For example:

    $tall(190) = 1$, $tall(185) = 0.8$, $tall(180) = 0.5$, $tall(160) = 0$.
- Something else than direct logic? Modelling? Machine learning? Machine teaching? Artificial Intelligence (AI)? Big Data? Convolutional Neural Networks (CNN)? Deep learning?

Welcome to Computer Vision and Pattern Recognition (CVPR) major!

Summary

- Sentences of a natural language can be formalized and represented mathematically as exact propositions in logic.
- Using propositional logic premises can produce new premises, truth values of propositions are defined, and conclusions are generated.
- Methods of logical reasoning:
  - The semantic method: the truth tables.
  - The syntactic method: the simplification of propositions and the rules of inference.
  - The resolution method.