Mathematics A Assignment 5 (week 41)

Group 1 (Thu 14/10, 12–14), Group 2 (Fri 15/10, 10–12), Group 3 (Fri 15/10, 12–14)

1. ($\Rightarrow$) Let $A \subseteq B$. Assume for contradiciton that

$$A \setminus B = \{a \in U \mid a \in A \text{ and } a \notin B\} \neq \emptyset.$$

This means that there is $x \in A$ such that $x \notin B$. This is not possible, because $A \subseteq B$, a contradiction. Therefore, $A \setminus B = \emptyset$.

($\Leftarrow$) Assume $A \setminus B = \emptyset$. By the definition of the set difference, this means that there is no element $x \in U$ such that $x \in A$ and $x \notin B$. Hence, if $a \in A$, then $a \in B$ and $A \subseteq B$.

2. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Which of the following relations are (i) functions, (ii) injections, (iii) surjections, (iv) bijections?

(a) $R_1 = \{(1, a), (2, c), (3, b), (4, d)\}$ is a bijection (all properties i-iv)

(b) $R_2 = \{(1, a), (2, b), (3, c), (4, c)\}$ is a function. Not injection, because 3 and 4 have the same image. Not surjection, because $d \in B$ is not an image of any element of A.

(c) $R_3 = \{(1, a), (2, b), (3, c), (3, d)\}$ is not a function, because $3 \in A$ is related to two elements in B.

3.

(a) $$2^{5x-2} = 16 \Leftrightarrow \log_2(2^{5x-2}) = \log_2 16 \Leftrightarrow 5x - 2 = 4 \Leftrightarrow 5x = 6 \Leftrightarrow x = \frac{6}{5}.$$

(b) $$5 \log_7 x = 10 \Leftrightarrow \log_7 x = 2 \Leftrightarrow 7^{\log_7 x} = 7^2 \Leftrightarrow x = 49.$$

(c) $$\log_2(3x - 7) = 5 \Leftrightarrow 2^{\log_2(3x-7)} = 2^5 \Leftrightarrow 3x - 7 = 32 \Leftrightarrow 3x = 39 \Leftrightarrow x = 13.$$

(b) $$\log_4 x + \log_4(x - 6) = 2 \Leftrightarrow \log_4(x(x - 6)) = 2 \Leftrightarrow \log_4(x^2 - 6x) = 2$$
$$\Leftrightarrow 4^{\log_4(x^2-6x)} = 4^2 \Leftrightarrow x^2 - 6x = 16 \Leftrightarrow x^2 - 6x - 16 = 0.$$

The last second degree polynomial has solutions $x = 8$ and $x = -2$, but $x = -2$ is not possible, because the “input” for logarithm-function needs to be positive. The only solution is therefore $x = 8$.

4. Suppose for the contradiction that there are integers x and y such that $x^2 = 4y+2$. Now $4y+2 = 2(2y+1)$ is an even number. This gives that x^2 is an even number and therefore (lectures!) x is an even number. There is an integer k such that $x = 2k$ and we have $x^2 = (2k)^2 = 4k^2 = 2(2y + 1)$. After division by 2, the last equality gives

$$2k^2 = 2y + 1.$$

This is impossible, because $2k^2$ is even and $2y + 1$ is odd, a contradiction! So, there are **no** integers x and y such that $x^2 = 4y + 2$.

5. We prove the contrapositive:

$$\neg(n \text{ is odd}) \implies \neg(5n \text{ is odd}),$$

that is,

$$n \text{ is even} \implies 5n \text{ is even}.$$

Assume n is even. This means that there is an integer k such that $n = 2k$. Now

$$5n = 5 \cdot 2k = 2 \cdot 5k$$

is also even.

6. Suppose for contradiction that $\log_{10}(7)$ is rational. This means that there are integers a and $b \neq 0$ such that

$$\log_{10}(7) = \frac{a}{b}.$$

Because $\log_{10}(7) > 0$, we may assume that a and b are both positive integers. We have that $b \log_{10}(7) = a$ and $a = \log_{10}(7^b)$. This also gives

$$10^a = 10^{\log_{10}(7^b)} = 7^b.$$

Now

$$10^a = 2 \cdot 5 \cdot 10^{a-1},$$

which means that 10^a is even. Becuase 7 is odd, 7^b is odd. This is a contradiction (a number cannot be odd and even). Thus, $\log_{10}(7)$ is irrational.