1.

$$\binom{70}{5} = \frac{70!}{5!\,65!} = \frac{66 \cdot 67 \cdot 68 \cdot 69 \cdot 70}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 11 \cdot 67 \cdot 17 \cdot 69 \cdot 14 = 12103014$$

$$\binom{121}{115} = \frac{121!}{115!\,6!} = \frac{116 \cdot 117 \cdot 118 \cdot 119 \cdot 120 \cdot 121}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 116 \cdot 39 \cdot 59 \cdot 119 \cdot 121 = 3843323484.$$

2.

$$\begin{aligned}(1+x)^7 &= \binom{7}{0}x^7 + \binom{7}{1}x^6 + \binom{7}{2}x^5 + \binom{7}{3}x^4 + \binom{7}{4}x^3 + \binom{7}{5}x^2 + \binom{7}{6}x^1 + \binom{7}{7}x^0 \\ &= x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^2 + 21x^2 + 7x + 1\end{aligned}$$

3. There are several ways to enumerate

$$F(\mathbb{N}) = \{X \subseteq \mathbb{N} \mid X \text{ is finite}\}.$$

(a) If $X = \{x_1, x_2, \ldots, x_n\}$ is a finite subset of $\mathbb{N}$, then the sum $x_1 + \cdots + x_n$ of its elements is a natural number. It is also clear that for each integer $n \in \mathbb{N}$, the number of sets X such that sum of the elements of X equals n is finite: each such set belongs to $\wp(\{0,1,2,\ldots,n\})$, whose size is finite. Therefore, we start with $\emptyset$, then enumerate all sets whose sum of elements is 0: $\{0\}$; then we enumerate all sets whose sum of elements is 1: $\{0,1\}$, $\{1\}$, then all sets whose sum is 2: $\{0,2\}$, $\{2\}$, sets whose sum is 3: $\{0,1,2\}$, $\{0,3\}, \{3\}$, and so.

Because each finite set is such that the sum of its elements is a natural number, each set is enumerated a some point. Also because the number of sets X such that sum of the elements of X equals n is finite, we never get stuck.

(b) Both *finite subsets of* $\mathbb{N}$ and *natural numbers* can be encoded as finite-length binary vectors. For instance, the binary representation of 6 is 110. This corresponds the set $\{1,2\}$ – the idea is that the rightmost bit corresponds to 0, second bit from right corresponds to 1, 3rd bit corresponds to 2, etc. The following is a bijection between finite sets and numbers:

$0 \leftrightarrow \emptyset$	$6 \leftrightarrow 110 \leftrightarrow \{1,2\}$	$12 \leftrightarrow 1100 \leftrightarrow \{2,3\}$
$1 \leftrightarrow \{0\}$	$7 \leftrightarrow 111 \leftrightarrow \{0,1,2\}$	$13 \leftrightarrow 1101 \leftrightarrow \{0,2,3\}$
$2 \leftrightarrow 10 \leftrightarrow \{1\}$	$8 \leftrightarrow 1000 \leftrightarrow \{3\}$	$14 \leftrightarrow 1110 \leftrightarrow \{1,2,3\}$
$3 \leftrightarrow 11 \leftrightarrow \{0,1\}$	$9 \leftrightarrow 1001 \leftrightarrow \{0,3\}$	$15 \leftrightarrow 1111 \leftrightarrow \{0,1,2,3\}$
$4 \leftrightarrow 100 \leftrightarrow \{2\}$	$10 \leftrightarrow 1010 \leftrightarrow \{1,3\}$	$16 \leftrightarrow 10000 \leftrightarrow \{4\}$
$5 \leftrightarrow 101 \leftrightarrow \{0,2\}$	$11 \leftrightarrow 1011 \leftrightarrow \{0,1,3\}$	$17 \leftrightarrow 10001 \leftrightarrow \{0,4\}$

4. The map $f\colon \mathbb{Z} \to \mathbb{N}$ is defined by

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n-1 & \text{if } n < 0 \end{cases}$$

Surjection: Let $n \in \mathbb{N}$. If n is *even*, then $n = 2k$ for some integer $k \geq 0$. We have $k = \dfrac{n}{2}$. Now $f(k) = 2 \cdot \dfrac{n}{2} = n$. If n is *odd*, then $n = 2k - 1$ for some integer $k \geq 1$. Now $k = \dfrac{n+1}{2}$ and $-k = \dfrac{-n-1}{2}$. Because $k \geq 1$, $-k < 0$. We have that $f(-k) = -2 \cdot \dfrac{-n-1}{2} - 1 = n$.

Injection: If $n \geq 0$, then $f(n)$ is even and if $n < 0$, then $f(n)$ is odd. The means that if $f(n_1) = f(n_2)$, we have only two cases:

(i) $n_1 \geq 0$ and $n_2 \geq 0$: $f(n_1) = f(n_2)$ implies $2n_1 = 2n_2$ and $n_1 = n_2$.
(ii) $n_1 < 0$ and $n_2 < 0$: $f(n_1) = f(n_2)$ implies $-2n_1 - 1 = -2n_2 - 1$ and $n_1 = n_2$.

Because f is injective and surjective, it is a bijection.

5. The map $f\colon (0,1) \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \dfrac{1}{x} - 2 & \text{if } 0 < x \leq \dfrac{1}{2} \\ \dfrac{1}{x-1} + 2 & \text{if } \dfrac{1}{2} < x < 1 \end{cases}$$

Surjection: Let $y \in \mathbb{R}$. If $y \geq 0$, then we set $y = \dfrac{1}{x} - 2$. This gives $\dfrac{1}{x} = y + 2$ and $x = \dfrac{1}{y+2}$. Now $0 < x \leq \dfrac{1}{2}$. We have $f(x) = y + 2 - 2 = y$. If $y < 0$, then we set $y = \dfrac{1}{x-1} + 2$. We have $\dfrac{1}{x-1} = y - 2$ and $x = \dfrac{1}{y-2} + 1 = \dfrac{y-1}{y-2}$. Now $\dfrac{1}{2} < x < 1$ and $f(x) = y$.

Injection: Let us first note that if $0 < x \leq \dfrac{1}{2}$, then $f(x)$ is positive and if $\dfrac{1}{2} < x < 1$, then $f(x)$ is negative. This means that if $f(x) = f(y)$, we have only two possibilities:

(i) $0 < x, y \leq \dfrac{1}{2}$: If $f(x) = f(y)$, then $\dfrac{1}{x} - 2 = \dfrac{1}{y} - 2$, which is equivalent to $x = y$.

(ii) $\dfrac{1}{2} < x, y < 1$: If $f(x) = f(y)$, then $\dfrac{1}{x-1} + 2 = \dfrac{1}{y-1} + 2$ gives $x = y$.

Because f is bijective, $|(0,1)| = |\mathbb{R}|$.

6. We prove that the are injections $f\colon (0,1) \times (0,1) \to (0,1)$ and $g\colon (0,1) \to (0,1) \to (0,1) \times (0,1)$.
(Injection f): Let $a \in (0,1)$. Then the map $f(x) = (a, x)$ is an injection $(0,1) \to (0,1) \times (0,1)$. Suppose that we have selected to represent real numbers so that the tail-end consists of 9's is excluded. Let

$$x = (0.a_1a_2a_3a_4a_5\cdots, 0.b_1b_2b_3b_4b_5\cdots) \in (0,1) \times (0,1).$$

(Injection g): Let us define $g(x)$ so that it is a number formed by taking decimal from the first 'coordinate' and 'second coordinate' one-by-one, that is,

$$g(x) = 0.a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5\cdots$$

Now clearly $g(x) \in (0,1)$. The map g is an injection, because if

$$f(x) = 0.a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5\cdots$$
$$f(y) = 0.c_1d_1c_2d_2c_3d_3c_4d_4c_5d_5\cdots$$

then $a_i = c_i$ and $b_i = d_i$ for all $i \geq 0$. We obtain

$$x = (0.a_1a_2a_3a_4a_5\cdots, 0.b_1b_2b_3b_4b_5\cdots)$$
$$y = (0.c_1c_2c_3c_4c_5\cdots, 0.d_1d_2d_3d_4d_5\cdots)$$

We have that $f\colon (0,1) \times (0,1) \to (0,1)$ and $g\colon (0,1) \times (0,1) \to (0,1)$ are injections. By **Schröder–Bernstein theorem**, $|(0,1) \times (0,1)| = |(0,1)|$.
Because $\mathbb{C} = |\mathbb{R} \times \mathbb{R}| = |(0,1) \times (0,1)| = |(0,1)| = |\mathbb{R}|$, the claim is proved.