1. Let A be the following augmented matrix

$$\left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right] \underset{(a)}{\Rightarrow} \left[\begin{array}{rrr|r} 20 & -5 & 15 & 25 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right] \underset{(b)}{\Rightarrow} \left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ -6 & 1 & -3 & 10 \\ 0 & 2 & 5 & 9 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right] \underset{(b)}{\Rightarrow} \left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ 12 & -1 & 14 & 24 \\ -6 & 1 & -3 & 10 \end{array}\right]$$

2. The biggest difficulty in this exercise is that one needs to be **very** careful in details. Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$. We set $AB = (c_{ij})_{m\times p}$. Therefore,

$$c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{in} \cdot b_{nj}$$

and

$$c_{ji} = a_{j1} \cdot b_{1i} + a_{j2} \cdot b_{2i} + \cdots + a_{jn} \cdot b_{ni}$$

Let $(d_{ij})_{p\times m} = (AB)^{\mathrm{T}}$. We have that $d_{ij} = c_{ji}$. On the other hand, let

$$B^{\mathrm{T}} = (e_{ij})_{p\times n}, \quad A^{\mathrm{T}} = (f_{ij})_{n\times m}, \quad B^{\mathrm{T}}A^{\mathrm{T}} = (g_{ij})_{p\times n}.$$

This means that $e_{ij} = b_{ji}$ and $f_{ij} = a_{ji}$. Now

$$\begin{aligned} g_{ij} &= e_{i1} \cdot f_{1j} + e_{i2} \cdot f_{2j} + \cdots e_{in} \cdot f_{nj} \\ &= b_{1i} \cdot a_{j1} + b_{2i} \cdot a_{j2} + \cdots b_{ni} \cdot a_{jn} \\ &= a_{j1} \cdot b_{1i} + a_{j2} \cdot b_{2i} + \cdots + a_{jn} \cdot b_{ni}. \end{aligned}$$

We have that $d_{ij} = g_{ij}$. Thus, $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$.

3. For

$$A = \begin{bmatrix} 10 & 0 & -3 \\ -2 & -4 & 1 \\ 3 & 0 & 2 \end{bmatrix},$$

$$\det A = 10 \cdot \det \begin{bmatrix} -4 & 1 \\ 0 & 2 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + c \cdot \det \begin{bmatrix} -2 & -4 \\ 3 & 0 \end{bmatrix}$$

Because $b = 0$, we do not need to compute the determinant of the "middle" case. We have

$$\det \begin{bmatrix} -4 & 1 \\ 0 & 2 \end{bmatrix} = -4 \cdot -0 = -8 \quad \text{and} \quad \det \begin{bmatrix} -2 & -4 \\ 3 & 0 \end{bmatrix} = 0 - 3 \cdot (-4) = 12$$

Thus, $\det A = 10 \cdot (-8) - 0 + (-3) \cdot 12 = -80 - 36 = -116$.

4. There are several paths which lead to the unique reduced row echelon form. This is one possibility:

$$\begin{bmatrix} 7 & -8 & -12 \\ -4 & 2 & 3 \end{bmatrix} \overset{2R_2+R_1\to R_2}{\Rightarrow} \begin{bmatrix} -1 & -4 & -6 \\ -4 & 2 & 3 \end{bmatrix} \overset{-1R_1\to R_1}{\Rightarrow} \begin{bmatrix} 1 & 4 & 6 \\ -4 & 2 & 3 \end{bmatrix} \overset{4R_1+R_2\to R_2}{\Rightarrow}$$

$$\begin{bmatrix} 1 & 4 & 6 \\ 0 & 18 & 27 \end{bmatrix} \overset{\frac{1}{18}R_2\to R_2}{\Rightarrow} \begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & \frac{3}{2} \end{bmatrix} \overset{-4R_2+R_1\to R_1}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \end{bmatrix}$$

This gives the solution $x = 0$ and $y = \frac{3}{2}$.

5. We reduce A to I_3:

$$\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 2 & 3 & -4\end{bmatrix} \overset{-R_1+R_2\to R_2}{\Longrightarrow} \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 2 & 3 & -4\end{bmatrix} \overset{-2R_1+R_3\to R_3}{\Longrightarrow} \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 5 & -4\end{bmatrix} \overset{-5R_2+R_3\to R_3}{\Longrightarrow} \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$$

$$\overset{R_3+R_2\to R_2}{\Longrightarrow} \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \overset{R_2+R_1\to R_1}{\Longrightarrow} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

We get A^{-1} by applying the above operations to I_3 in the same order:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \overset{-R_1+R_2\to R_2}{\Longrightarrow} \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \overset{-2R_1+R_3\to R_3}{\Longrightarrow} \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -2 & 0 & 1\end{bmatrix} \overset{-5R_2+R_3\to R_3}{\Longrightarrow} \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 3 & -5 & 1\end{bmatrix}$$

$$\overset{R_3+R_2\to R_2}{\Longrightarrow} \begin{bmatrix}1 & 0 & 0\\ 2 & -4 & 1\\ 3 & -5 & 1\end{bmatrix} \overset{R_2+R_1\to R_1}{\Longrightarrow} \begin{bmatrix}3 & -4 & 1\\ 2 & -4 & 1\\ 3 & -5 & 1\end{bmatrix}$$

The result can be verified by multiplying:

$$\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 2 & 3 & -4\end{bmatrix}\begin{bmatrix}3 & -4 & 1\\ 2 & -4 & 1\\ 3 & -5 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

By Proposition 18 this is enough to show that these two matrices are inverses.

6. (a)

$$M_R = \begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 1\end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}$$

(b) $M_{R\circ S} = M_R \circ M_S$:

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

(c) Let us first form $M_{R^{-1}} = (M_R)^{\mathrm{T}}$

$$\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$$

Now the matrix of $R \circ R^{-1}$ can be formed as the product

$$\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$