

## Chapter 5

# Set Theory

### 5.1 Sets and Operations on Sets

#### Preview Activity 1 (Set Operations)

Before beginning this section, it would be a good idea to review sets and set notation, including the roster method and set builder notation, in Section 2.3.

In Section 2.1, we used logical operators (conjunction, disjunction, negation) to form new statements from existing statements. In a similar manner, there are several ways to create new sets from sets that have already been defined. In fact, we will form these new sets using the logical operators of conjunction (and), disjunction (or), and negation (not). For example, if the universal set is the set of natural numbers  $\mathbb{N}$  and

$$A = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad B = \{1, 3, 5, 7, 9\},$$

- The set consisting of all natural numbers that are in  $A$  and are in  $B$  is the set  $\{1, 3, 5\}$ ;
- The set consisting of all natural numbers that are in  $A$  or are in  $B$  is the set  $\{1, 2, 3, 4, 5, 6, 7, 9\}$ ; and
- The set consisting of all natural numbers that are in  $A$  and are not in  $B$  is the set  $\{2, 4, 6\}$ .

These sets are examples of some of the most common set operations, which are given in the following definitions.

**Definition.** Let  $A$  and  $B$  be subsets of some universal set  $U$ . The **intersection** of  $A$  and  $B$ , written  $A \cap B$  and read “ $A$  intersect  $B$ ,” is the set of all elements that are in both  $A$  and  $B$ . That is,

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}.$$

The **union** of  $A$  and  $B$ , written  $A \cup B$  and read “ $A$  union  $B$ ,” is the set of all elements that are in  $A$  or in  $B$ . That is,

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}.$$

**Definition.** Let  $A$  and  $B$  be subsets of some universal set  $U$ . The **set difference** of  $A$  and  $B$ , or **relative complement** of  $B$  with respect to  $A$ , written  $A - B$  and read “ $A$  minus  $B$ ” or “the complement of  $B$  with respect to  $A$ ,” is the set of all elements in  $A$  that are not in  $B$ . That is,

$$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}.$$

The **complement** of the set  $A$ , written  $A^c$  and read “the complement of  $A$ ,” is the set of all elements of  $U$  that are not in  $A$ . That is,

$$A^c = \{x \in U \mid x \notin A\}.$$

For the rest of this preview activity, the universal set is  $U = \{0, 1, 2, 3, \dots, 10\}$ , and we will use the following subsets of  $U$ :

$$A = \{0, 1, 2, 3, 9\} \quad \text{and} \quad B = \{2, 3, 4, 5, 6\}.$$

So in this case,  $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\} = \{2, 3\}$ . Use the roster method to specify each of the following subsets of  $U$ .

1.  $A \cup B$

2.  $A^c$

3.  $B^c$

We can now use these sets to form even more sets. For example,

$$A \cap B^c = \{0, 1, 2, 3, 9\} \cap \{0, 1, 7, 8, 9, 10\} = \{0, 1, 9\}.$$

Use the roster method to specify each of the following subsets of  $U$ .



4.  $A \cup B^c$
5.  $A^c \cap B^c$
6.  $A^c \cup B^c$
7.  $(A \cap B)^c$

**Preview Activity 2 (Venn Diagrams for Two Sets)**

In Preview Activity 1, we worked with verbal and symbolic definitions of set operations. However, it is also helpful to have a visual representation of sets. **Venn diagrams** are used to represent sets by circles (or some other closed geometric shape) drawn inside a rectangle. The points inside the rectangle represent the universal set  $U$ , and the elements of a set are represented by the points inside the circle that represents the set. For example, Figure 5.1 is a Venn diagram showing two sets.

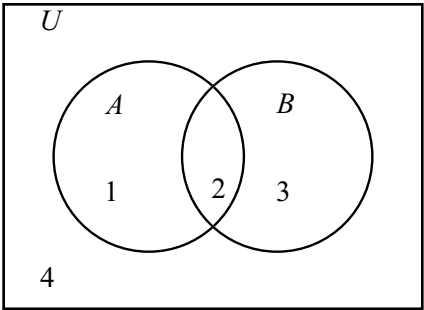


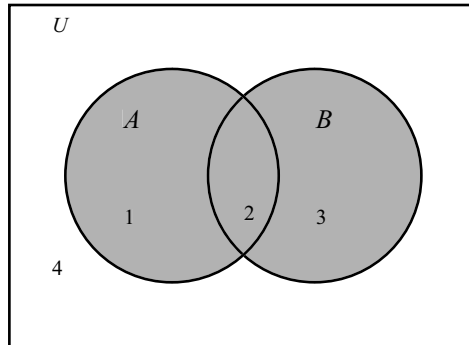
Figure 5.1: Venn Diagram for Two Sets

In Figure 5.1, the elements of  $A$  are represented by the points inside the left circle, and the elements of  $B$  are represented by the points inside the right circle. The four distinct regions in the diagram are numbered for reference purposes only. (The numbers do not represent elements in a set.) The following table describes the four regions in the diagram.

Region	Elements of $U$	Set
1	In $A$ and not in $B$	$A - B$
2	In $A$ and in $B$	$A \cap B$
3	In $B$ and not in $A$	$B - A$
4	Not in $A$ and not in $B$	$A^c \cap B^c$

We can use these regions to represent other sets. For example, the set  $A \cup B$  is represented by regions 1, 2, and 3 or the shaded region in Figure 5.2.



Figure 5.2: Venn Diagram for  $A \cup B$ 

Let  $A$  and  $B$  be subsets of a universal set  $U$ . For each of the following, draw a Venn diagram for two sets and shade the region that represent the specified set. In addition, describe the set using set builder notation.

1.  $A^c$

3.  $A^c \cup B$

5.  $(A \cap B)^c$

2.  $B^c$

4.  $A^c \cup B^c$

6.  $(A \cup B) - (A \cap B)$

### Set Equality, Subsets, and Proper Subsets

In Section 2.3, we introduced some basic definitions used in set theory, what it means to say that two sets are equal and what it means to say that one set is a subset of another set. See the definitions on page 55. We need one more definition.

**Definition.** Let  $A$  and  $B$  be two sets contained in some universal set  $U$ . The set  $A$  is a **proper subset** of  $B$  provided that  $A \subseteq B$  and  $A \neq B$ . When  $A$  is a proper subset of  $B$ , we write  $A \subset B$ .

One reason for the definition of proper subset is that each set is a subset of itself. That is,

$$\text{If } A \text{ is a set, then } A \subseteq A.$$

However, sometimes we need to indicate that a set  $X$  is a subset of  $Y$  but  $X \neq Y$ . For example, if

$$X = \{1, 2\} \text{ and } Y = \{0, 1, 2, 3\},$$



then  $X \subset Y$ . We know that  $X \subseteq Y$  since each element of  $X$  is an element of  $Y$ , but  $X \neq Y$  since  $0 \in Y$  and  $0 \notin X$ . (Also,  $3 \in Y$  and  $3 \notin X$ .) Notice that the notations  $A \subset B$  and  $A \subseteq B$  are used in a manner similar to inequality notation for numbers ( $a < b$  and  $a \leq b$ ).

It is often very important to be able to describe precisely what it means to say that one set is not a subset of the other. In the preceding example,  $Y$  is not a subset of  $X$  since there exists an element of  $Y$  (namely, 0) that is not in  $X$ .

In general, the subset relation is described with the use of a universal quantifier since  $A \subseteq B$  means that for each element  $x$  of  $U$ , if  $x \in A$ , then  $x \in B$ . So when we negate this, we use an existential quantifier as follows:

$$\begin{aligned} A \subseteq B & \quad \text{means} \quad (\forall x \in U) [(x \in A) \rightarrow (x \in B)]. \\ A \not\subseteq B & \quad \text{means} \quad \neg(\forall x \in U) [(x \in A) \rightarrow (x \in B)] \\ & \quad (\exists x \in U) \neg[(x \in A) \rightarrow (x \in B)] \\ & \quad (\exists x \in U) [(x \in A) \wedge (x \notin B)]. \end{aligned}$$

So we see that  $A \not\subseteq B$  means that there exists an  $x$  in  $U$  such that  $x \in A$  and  $x \notin B$ .

Notice that if  $A = \emptyset$ , then the conditional statement, “For each  $x \in U$ , if  $x \in \emptyset$ , then  $x \in B$ ” must be true since the hypothesis will always be false. Another way to look at this is to consider the following statement:

$$\emptyset \not\subseteq B \text{ means that there exists an } x \in \emptyset \text{ such that } x \notin B.$$

However, this statement must be false since there does not exist an  $x$  in  $\emptyset$ . Since this is false, we must conclude that  $\emptyset \subseteq B$ . Although the facts that  $\emptyset \subseteq B$  and  $B \subseteq B$  may not seem very important, we will use these facts later, and hence we summarize them in Theorem 5.1.

**Theorem 5.1.** *For any set  $B$ ,  $\emptyset \subseteq B$  and  $B \subseteq B$ .*

In Section 2.3, we also defined two sets to be equal when they have precisely the same elements. For example,

$$\{x \in \mathbb{R} \mid x^2 = 4\} = \{-2, 2\}.$$

If the two sets  $A$  and  $B$  are equal, then it must be true that every element of  $A$  is an element of  $B$ , that is,  $A \subseteq B$ , and it must be true that every element of  $B$  is



an element of  $A$ , that is,  $B \subseteq A$ . Conversely, if  $A \subseteq B$  and  $B \subseteq A$ , then  $A$  and  $B$  must have precisely the same elements. This gives us the following test for set equality:

**Theorem 5.2.** *Let  $A$  and  $B$  be subsets of some universal set  $U$ . Then  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ .*

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**Progress Check 5.3 (Using Set Notation)**

Let the universal set be  $U = \{1, 2, 3, 4, 5, 6\}$ , and let

$$A = \{1, 2, 4\}, \quad B = \{1, 2, 3, 5\}, \quad C = \{x \in U \mid x^2 \leq 2\}.$$

In each of the following, fill in the blank with one or more of the symbols  $\subset$ ,  $\subseteq$ ,  $=$ ,  $\neq$ ,  $\in$ , or  $\notin$  so that the resulting statement is true. For each blank, include all symbols that result in a true statement. If none of these symbols makes a true statement, write nothing in the blank.

$A$	_____	$B$		$\emptyset$	_____	$A$
5	_____	$B$		$\{5\}$	_____	$B$
$A$	_____	$C$		$\{1, 2\}$	_____	$C$
$\{1, 2\}$	_____	$A$		$\{4, 2, 1\}$	_____	$A$
6	_____	$A$		$B$	_____	$\emptyset$

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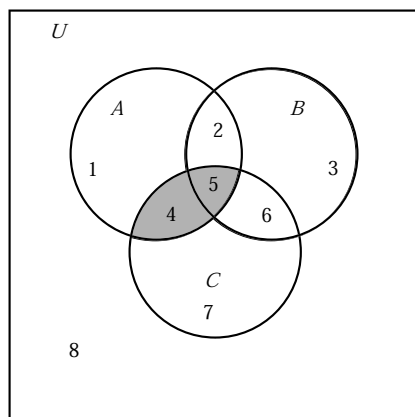
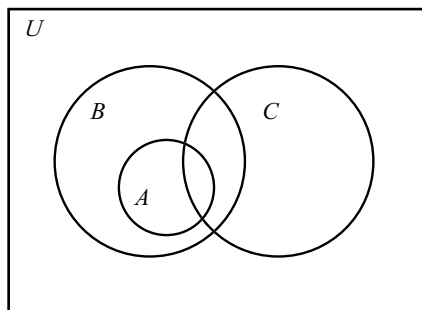
**More about Venn Diagrams**

In Preview Activity 2, we learned how to use Venn diagrams as a visual representation for sets, set operations, and set relationships. In that activity, we restricted ourselves to using two sets. We can, of course, include more than two sets in a Venn diagram. Figure 5.3 shows a general Venn diagram for three sets (including a shaded region that corresponds to  $A \cap C$ ).

In this diagram, there are eight distinct regions, and each region has a unique reference number. For example, the set  $A$  is represented by the combination of regions 1, 2, 4, and 5, whereas the set  $C$  is represented by the combination of regions 4, 5, 6, and 7. This means that the set  $A \cap C$  is represented by the combination of regions 4 and 5. This is shown as the shaded region in Figure 5.3.

Finally, Venn diagrams can also be used to illustrate special relationships between sets. For example, if  $A \subseteq B$ , then the circle representing  $A$  should be completely contained in the circle for  $B$ . So if  $A \subseteq B$ , and we know nothing about



Figure 5.3: Venn Diagram for  $A \cap C$ Figure 5.4: Venn Diagram Showing  $A \subseteq B$ 

any relationship between the set  $C$  and the sets  $A$  and  $B$ , we could use the Venn diagram shown in Figure 5.4.

#### Progress Check 5.4 (Using Venn Diagrams)

Let  $A$ ,  $B$ , and  $C$  be subsets of a universal set  $U$ .

- For each of the following, draw a Venn diagram for three sets and shade the region(s) that represent the specified set.

(a)  $(A \cap B) \cap C$

(c)  $(A^c \cup B)$

(b)  $(A \cap B) \cup C$

(d)  $A^c \cap (B \cup C)$

2. Draw the most general Venn diagram showing  $B \subseteq (A \cup C)$ .
3. Draw the most general Venn diagram showing  $A \subseteq (B^c \cup C)$ .

### The Power Set of a Set

The symbol  $\in$  is used to describe a relationship between an element of the universal set and a subset of the universal set, and the symbol  $\subseteq$  is used to describe a relationship between two subsets of the universal set. For example, the number 5 is an integer, and so it is appropriate to write  $5 \in \mathbb{Z}$ . It is not appropriate, however, to write  $5 \subseteq \mathbb{Z}$  since 5 is not a set. It is important to distinguish between 5 and  $\{5\}$ . The difference is that 5 is an integer and  $\{5\}$  is a set consisting of one element. Consequently, it is appropriate to write  $\{5\} \subseteq \mathbb{Z}$ , but it is not appropriate to write  $\{5\} \in \mathbb{Z}$ . The distinction between these two symbols (5 and  $\{5\}$ ) is important when we discuss what is called the power set of a given set.

**Definition.** If  $A$  is a subset of a universal set  $U$ , then the set whose members are all the subsets of  $A$  is called the **power set** of  $A$ . We denote the power set of  $A$  by  $\mathcal{P}(A)$ . Symbolically, we write

$$\mathcal{P}(A) = \{X \subseteq U \mid X \subseteq A\}.$$

That is,  $X \in \mathcal{P}(A)$  if and only if  $X \subseteq A$ .

When dealing with the power set of  $A$ , we must always remember that  $\emptyset \subseteq A$  and  $A \subseteq A$ . For example, if  $A = \{a, b\}$ , then the subsets of  $A$  are

$$\emptyset, \{a\}, \{b\}, \{a, b\}. \quad (1)$$

We can write this as

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Now let  $B = \{a, b, c\}$ . Notice that  $B = A \cup \{c\}$ . We can determine the subsets of  $B$  by starting with the subsets of  $A$  in (1). We can form the other subsets of  $B$  by taking the union of each set in (1) with the set  $\{c\}$ . This gives us the following subsets of  $B$ .

$$\{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}. \quad (2)$$

So the subsets of  $B$  are those sets in (1) combined with those sets in (2). That is, the subsets of  $B$  are

$$\emptyset, \{a\}, \{b\}, \{a, b\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \quad (3)$$





which means that

$$\mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

Notice that we could write

$$\{a, c\} \subseteq B \text{ or that } \{a, c\} \in \mathcal{P}(B).$$

Also, notice that  $A$  has two elements and  $A$  has four subsets, and  $B$  has three elements and  $B$  has eight subsets. Now, let  $n$  be a nonnegative integer. The following result can be proved using mathematical induction. (See Exercise 17.)

**Theorem 5.5.** *Let  $n$  be a nonnegative integer and let  $T$  be a subset of some universal set. If the set  $T$  has  $n$  elements, then the set  $T$  has  $2^n$  subsets. That is,  $\mathcal{P}(T)$  has  $2^n$  elements.*

## The Cardinality of a Finite Set

In our discussion of the power set, we were concerned with the number of elements in a set. In fact, the number of elements in a finite set is a distinguishing characteristic of the set, so we give it the following name.

**Definition.** The number of elements in a finite set  $A$  is called the **cardinality** of  $A$  and is denoted by  $\text{card}(A)$ .

For example,  $\text{card}(\emptyset) = 0$ ;       $\text{card}(\{a, b\}) = 2$ ;       $\text{card}(\mathcal{P}(\{a, b\})) = 4$ .

**Theoretical Note:** There is a mathematical way to distinguish between finite and infinite sets, and there is a way to define the cardinality of an infinite set. We will not concern ourselves with this at this time. More about the cardinality of finite and infinite sets is discussed in Chapter 9.

## Standard Number Systems

We can use set notation to specify and help describe our standard number systems. The starting point is the set of **natural numbers**, for which we use the roster method.

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$



The **integers** consist of the natural numbers, the negatives of the natural numbers, and zero. If we let  $\mathbb{N}^- = \{\dots, -4, -3, -2, -1\}$ , then we can use set union and write

$$\mathbb{Z} = \mathbb{N}^- \cup \{0\} \cup \mathbb{N}.$$

So we see that  $\mathbb{N} \subseteq \mathbb{Z}$ , and in fact,  $\mathbb{N} \subset \mathbb{Z}$ .

We need to use set builder notation for the set  $\mathbb{Q}$  of all **rational numbers**, which consists of quotients of integers.

$$\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}$$

Since any integer  $n$  can be written as  $n = \frac{n}{1}$ , we see that  $\mathbb{Z} \subseteq \mathbb{Q}$ .

We do not yet have the tools to give a complete description of the real numbers. We will simply say that the **real numbers** consist of the rational numbers and the **irrational numbers**. In effect, the irrational numbers are the complement of the set of rational numbers  $\mathbb{Q}$  in  $\mathbb{R}$ . So we can use the notation  $\mathbb{Q}^c = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$  and write

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c \quad \text{and} \quad \mathbb{Q} \cap \mathbb{Q}^c = \emptyset.$$

A number system that we have not yet discussed is the set of **complex numbers**. The complex numbers,  $\mathbb{C}$ , consist of all numbers of the form  $a + bi$ , where  $a, b \in \mathbb{R}$  and  $i = \sqrt{-1}$  (or  $i^2 = -1$ ). That is,

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}.$$

We can add and multiply complex numbers as follows: If  $a, b, c, d \in \mathbb{R}$ , then

$$\begin{aligned} (a + bi) + (c + di) &= (a + c) + (b + d)i, \text{ and} \\ (a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

## Exercises for Section 5.1

\* 1. Assume the universal set is the set of real numbers. Let

$$\begin{aligned} A &= \{-3, -2, 2, 3\}, & B &= \{x \in \mathbb{R} \mid x^2 = 4 \text{ or } x^2 = 9\}, \\ C &= \{x \in \mathbb{R} \mid x^2 + 2 = 0\}, & D &= \{x \in \mathbb{R} \mid x > 0\}. \end{aligned}$$

Respond to each of the following questions. In each case, explain your answer.



- (a) Is the set  $A$  equal to the set  $B$ ?  
 (b) Is the set  $A$  a subset of the set  $B$ ?  
 (c) Is the set  $C$  equal to the set  $D$ ?  
 (d) Is the set  $C$  a subset of the set  $D$ ?  
 (e) Is the set  $A$  a subset of the set  $D$ ?
- \* 2. (a) Explain why the set  $\{a, b\}$  is equal to the set  $\{b, a\}$ .  
 (b) Explain why the set  $\{a, b, b, a, c\}$  is equal to the set  $\{b, c, a\}$ .
- \* 3. Assume that the universal set is the set of integers. Let

$$\begin{aligned} A &= \{-3, -2, 2, 3\}, & B &= \{x \in \mathbb{Z} \mid x^2 \leq 9\}, \\ C &= \{x \in \mathbb{Z} \mid x \geq -3\}, & D &= \{1, 2, 3, 4\}. \end{aligned}$$

In each of the following, fill in the blank with one or more of the symbols  $\subset$ ,  $\subseteq$ ,  $\not\subseteq$ ,  $=$ ,  $\neq$ ,  $\in$ , or  $\notin$  so that the resulting statement is true. For each blank, include all symbols that result in a true statement. If none of these symbols makes a true statement, write nothing in the blank.

$A$	_____	$B$	$\emptyset$	_____	$A$
5	_____	$C$	$\{5\}$	_____	$C$
$A$	_____	$C$	$\{1, 2\}$	_____	$B$
$\{1, 2\}$	_____	$A$	$\{3, 2, 1\}$	_____	$D$
4	_____	$B$	$D$	_____	$\emptyset$
$\text{card}(A)$	_____	$\text{card}(D)$	$\text{card}(A)$	_____	$\text{card}(B)$
$A$	_____	$\mathcal{P}(A)$	$A$	_____	$\mathcal{P}(B)$

- \* 4. Write all of the proper subset relations that are possible using the sets of numbers  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$ .
- \* 5. For each statement, write a brief, clear explanation of why the statement is true or why it is false.
- (a) The set  $\{a, b\}$  is a subset of  $\{a, c, d, e\}$ .  
 (b) The set  $\{-2, 0, 2\}$  is equal to  $\{x \in \mathbb{Z} \mid x \text{ is even and } x^2 < 5\}$ .  
 (c) The empty set  $\emptyset$  is a subset of  $\{1\}$ .  
 (d) If  $A = \{a, b\}$ , then the set  $\{a\}$  is a subset of  $\mathcal{P}(A)$ .

6. Use the definitions of set intersection, set union, and set difference to write useful negations of these definitions. That is, complete each of the following sentences

- \* (a)  $x \notin A \cap B$  if and only if ....  
 (b)  $x \notin A \cup B$  if and only if ....  
 (c)  $x \notin A - B$  if and only if ....

- \* 7. Let  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ , and let

$$\begin{aligned} A &= \{3, 4, 5, 6, 7\}, & B &= \{1, 5, 7, 9\}, \\ C &= \{3, 6, 9\}, & D &= \{2, 4, 6, 8\}. \end{aligned}$$

Use the roster method to list all of the elements of each of the following sets.

- |                         |                                  |
|-------------------------|----------------------------------|
| (a) $A \cap B$          | (h) $(A \cap C) \cup (B \cap C)$ |
| (b) $A \cup B$          | (i) $B \cap D$                   |
| (c) $(A \cup B)^c$      | (j) $(B \cap D)^c$               |
| (d) $A^c \cap B^c$      | (k) $A - D$                      |
| (e) $(A \cup B) \cap C$ | (l) $B - D$                      |
| (f) $A \cap C$          | (m) $(A - D) \cup (B - D)$       |
| (g) $B \cap C$          | (n) $(A \cup B) - D$             |

8. Let  $U = \mathbb{N}$ , and let

$$\begin{aligned} A &= \{x \in \mathbb{N} \mid x \geq 7\}, & B &= \{x \in \mathbb{N} \mid x \text{ is odd}\}, \\ C &= \{x \in \mathbb{N} \mid x \text{ is a multiple of } 3\}, & D &= \{x \in \mathbb{N} \mid x \text{ is even}\}. \end{aligned}$$

Use the roster method to list all of the elements of each of the following sets.

- |                                  |                            |
|----------------------------------|----------------------------|
| (a) $A \cap B$                   | (g) $B \cap D$             |
| (b) $A \cup B$                   | (h) $(B \cap D)^c$         |
| (c) $(A \cup B)^c$               | (i) $A - D$                |
| (d) $A^c \cap B^c$               | (j) $B - D$                |
| (e) $(A \cup B) \cap C$          | (k) $(A - D) \cup (B - D)$ |
| (f) $(A \cap C) \cup (B \cap C)$ | (l) $(A \cup B) - D$       |

9. Let  $P$ ,  $Q$ ,  $R$ , and  $S$  be subsets of a universal set  $U$ . Assume that  $(P - Q) \subseteq (R \cap S)$ .

(a) Complete the following sentence:

For each  $x \in U$ , if  $x \in (P - Q)$ , then . . . .

- \* (b) Write a useful negation of the statement in Part (9a).  
 (c) Write the contrapositive of the statement in Part (9a).

10. Let  $U$  be the universal set. Consider the following statement:

For all  $A$  and  $B$  that are subsets of  $U$ , if  $A \subseteq B$ , then  $B^c \subseteq A^c$ .

- \* (a) Identify three conditional statements in the given statement.  
 (b) Write the contrapositive of this statement.  
 (c) Write the negation of this statement.

11. Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ . Draw a Venn diagram for each of the following situations.

- (a)  $A \subseteq C$   
 (b)  $A \cap B = \emptyset$   
 (c)  $A \not\subseteq B$ ,  $B \not\subseteq A$ ,  $C \subseteq A$ , and  $C \not\subseteq B$   
 (d)  $A \subseteq B$ ,  $C \subseteq B$ , and  $A \cap C = \emptyset$

12. Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ . For each of the following, draw a general Venn diagram for the three sets and then shade the indicated region.

- |                                  |                         |
|----------------------------------|-------------------------|
| (a) $A \cap B$                   | (d) $B \cup C$          |
| (b) $A \cap C$                   | (e) $A \cap (B \cup C)$ |
| (c) $(A \cap B) \cup (A \cap C)$ | (f) $(A \cap B) - C$    |

13. We can extend the idea of consecutive integers (See Exercise (10) in Section 3.5) to represent four consecutive integers as  $m$ ,  $m + 1$ ,  $m + 2$ , and  $m + 3$ , where  $m$  is an integer. There are other ways to represent four consecutive integers. For example, if  $k \in \mathbb{Z}$ , then  $k - 1$ ,  $k$ ,  $k + 1$ , and  $k + 2$  are four consecutive integers.

- (a) Prove that for each  $n \in \mathbb{Z}$ ,  $n$  is the sum of four consecutive integers if and only if  $n \equiv 2 \pmod{4}$ .



- (b) Use set builder notation or the roster method to specify the set of integers that are the sum of four consecutive integers.
  - (c) Specify the set of all natural numbers that can be written as the sum of four consecutive natural numbers.
  - (d) Prove that for each  $n \in \mathbb{Z}$ ,  $n$  is the sum of eight consecutive integers if and only if  $n \equiv 4 \pmod{8}$ .
  - (e) Use set builder notation or the roster method to specify the set of integers that are the sum of eight consecutive integers.
  - (f) Specify the set of all natural numbers can be written as the sum of eight consecutive natural numbers.
14. One of the properties of real numbers is the so-called **Law of Trichotomy**, which states that if  $a, b \in \mathbb{R}$ , then exactly one of the following is true:

- $a < b$ ;
- $a = b$ ;
- $a > b$ .

Is the following proposition concerning sets true or false? Either provide a proof that it is true or a counterexample showing it is false.

If  $A$  and  $B$  are subsets of some universal set, then exactly one of the following is true:

- $A \subseteq B$ ;
- $A = B$ ;
- $B \subseteq A$ .

## Explorations and Activities

15. **Intervals of Real Numbers.** In previous mathematics courses, we have frequently used subsets of the real numbers called **intervals**. There are some common names and notations for intervals. These are given in the following table, where it is assumed that  $a$  and  $b$  are real numbers and  $a < b$ .

Interval		Name
Notation	Set Notation	
$(a, b) =$	$\{x \in \mathbb{R} \mid a < x < b\}$	Open interval from $a$ to $b$
$[a, b] =$	$\{x \in \mathbb{R} \mid a \leq x \leq b\}$	Closed interval from $a$ to $b$
$[a, b) =$	$\{x \in \mathbb{R} \mid a \leq x < b\}$	Half-open interval
$(a, b] =$	$\{x \in \mathbb{R} \mid a < x \leq b\}$	Half-open interval
$(a, +\infty) =$	$\{x \in \mathbb{R} \mid x > a\}$	Open ray
$(-\infty, b) =$	$\{x \in \mathbb{R} \mid x < b\}$	Open ray
$[a, +\infty) =$	$\{x \in \mathbb{R} \mid x \geq a\}$	Closed ray
$(-\infty, b] =$	$\{x \in \mathbb{R} \mid x \leq b\}$	Closed ray



- (a) Is  $(a, b)$  a proper subset of  $(a, b]$ ? Explain.
- (b) Is  $[a, b]$  a subset of  $(a, +\infty)$ ? Explain.
- (c) Use interval notation to describe
  - i. the intersection of the interval  $[-3, 7]$  with the interval  $(5, 9]$ ;
  - ii. the union of the interval  $[-3, 7]$  with the interval  $(5, 9]$ ;
  - iii. the set difference  $[-3, 7] - (5, 9]$ .
- (d) Write the set  $\{x \in \mathbb{R} \mid |x| \leq 0.01\}$  using interval notation.
- (e) Write the set  $\{x \in \mathbb{R} \mid |x| > 2\}$  as the union of two intervals.

**16. More Work with Intervals.** For this exercise, use the interval notation described in Exercise 15.

- (a) Determine the intersection and union of  $[2, 5]$  and  $[-1, +\infty)$ .
- (b) Determine the intersection and union of  $[2, 5]$  and  $[3.4, +\infty)$ .
- (c) Determine the intersection and union of  $[2, 5]$  and  $[7, +\infty)$ .

Now let  $a$ ,  $b$ , and  $c$  be real numbers with  $a < b$ .

- (d) Explain why the intersection of  $[a, b]$  and  $[c, +\infty)$  is either a closed interval, a set with one element, or the empty set.
- (e) Explain why the union of  $[a, b]$  and  $[c, +\infty)$  is either a closed ray or the union of a closed interval and a closed ray.

**17. Proof of Theorem 5.5.** To help with the proof by induction of Theorem 5.5, we first prove the following lemma. (The idea for the proof of this lemma was illustrated with the discussion of power set after the definition on page 222.)

**Lemma 5.6.** *Let  $A$  and  $B$  be subsets of some universal set. If  $A = B \cup \{x\}$ , where  $x \notin B$ , then any subset of  $A$  is either a subset of  $B$  or a set of the form  $C \cup \{x\}$ , where  $C$  is a subset of  $B$ .*

**Proof.** Let  $A$  and  $B$  be subsets of some universal set, and assume that  $A = B \cup \{x\}$  where  $x \notin B$ . Let  $Y$  be a subset of  $A$ . We need to show that  $Y$  is a subset of  $B$  or that  $Y = C \cup \{x\}$ , where  $C$  is some subset of  $B$ . There are two cases to consider: (1)  $x$  is not an element of  $Y$ , and (2)  $x$  is an element of  $Y$ .

*Case 1:* Assume that  $x \notin Y$ . Let  $y \in Y$ . Then  $y \in A$  and  $y \neq x$ . Since

$$A = B \cup \{x\},$$



this means that  $y$  must be in  $B$ . Therefore,  $Y \subseteq B$ .

*Case 2:* Assume that  $x \in Y$ . In this case, let  $C = Y - \{x\}$ . Then every element of  $C$  is an element of  $B$ . Hence, we can conclude that  $C \subseteq B$  and that  $Y = C \cup \{x\}$ .

Cases (1) and (2) show that if  $Y \subseteq A$ , then  $Y \subseteq B$  or  $Y = C \cup \{x\}$ , where  $C \subseteq B$ . ■

To begin the induction proof of Theorem 5.5, for each nonnegative integer  $n$ , we let  $P(n)$  be, “If a finite set has exactly  $n$  elements, then that set has exactly  $2^n$  subsets.”

- (a) Verify that  $P(0)$  is true. (This is the basis step for the induction proof.)
- (b) Verify that  $P(1)$  and  $P(2)$  are true.
- (c) Now assume that  $k$  is a nonnegative integer and assume that  $P(k)$  is true. That is, assume that if a set has  $k$  elements, then that set has  $2^k$  subsets. (This is the inductive assumption for the induction proof.)

Let  $T$  be a subset of the universal set with  $\text{card}(T) = k + 1$ , and let  $x \in T$ . Then the set  $B = T - \{x\}$  has  $k$  elements.

Now use the inductive assumption to determine how many subsets  $B$  has. Then use Lemma 5.6 to prove that  $T$  has twice as many subsets as  $B$ . This should help complete the inductive step for the induction proof.

## 5.2 Proving Set Relationships

### Preview Activity 1 (Working with Two Specific Sets)

Let  $S$  be the set of all integers that are multiples of 6, and let  $T$  be the set of all even integers.

1. List at least four different positive elements of  $S$  and at least four different negative elements of  $S$ . Are all of these integers even?
2. Use the roster method to specify the sets  $S$  and  $T$ . (See Section 2.3 for a review of the roster method.) Does there appear to be any relationship between these two sets? That is, does it appear that the sets are equal or that one set is a subset of the other set?





3. Use set builder notation to specify the sets  $S$  and  $T$ . (See Section 2.3 for a review of the set builder notation.)
4. Using appropriate definitions, describe what it means to say that an integer  $x$  is a multiple of 6 and what it means to say that an integer  $y$  is even.
5. In order to prove that  $S$  is a subset of  $T$ , we need to prove that for each integer  $x$ , if  $x \in S$ , then  $x \in T$ .

Complete the know-show table in Table 5.1 for the proposition that  $S$  is a subset of  $T$ .

This table is in the form of a proof method called the **choose-an-element method**. This method is frequently used when we encounter a universal quantifier in a statement in the backward process. (In this case, this is Step  $Q1$ .) The key is that we have to prove something about all elements in  $\mathbb{Z}$ . We can then add something to the forward process by choosing an arbitrary element from the set  $S$ . (This is done in Step  $P1$ .) This does not mean that we can choose a specific element of  $S$ . Rather, we must give the arbitrary element a name and use only the properties it has by being a member of the set  $S$ . In this case, the element is a multiple of 6.

Step	Know	Reason
$P$	$S$ is the set of all integers that are multiples of 6. $T$ is the set of all even integers.	Hypothesis
$P1$	Let $x \in S$ .	Choose an arbitrary element of $S$ .
$P2$	$(\exists m \in \mathbb{Z}) (x = 6m)$	Definition of “multiple”
$\vdots$	$\vdots$	$\vdots$
$Q2$	$x$ is an element of $T$ .	$x$ is even
$Q1$	$(\forall x \in \mathbb{Z}) [(x \in S) \rightarrow (x \in T)]$	Step $P1$ and Step $Q2$
$Q$	$S \subseteq T$ .	Definition of “subset”
Step	Show	Reason

Table 5.1: Know-show table for Preview Activity 1

### Preview Activity 2 (Working with Venn Diagrams)

1. Draw a Venn diagram for two sets,  $A$  and  $B$ , with the assumption that  $A$  is a subset of  $B$ . On this Venn diagram, lightly shade the area corresponding to  $A^c$ . Then, determine the region on the Venn diagram that corresponds to  $B^c$ . What appears to be the relationship between  $A^c$  and  $B^c$ ? Explain.
2. Draw a general Venn diagram for two sets,  $A$  and  $B$ . First determine the region that corresponds to the set  $A - B$  and then, on the Venn diagram, shade the region corresponding to  $A - (A - B)$  and shade the region corresponding to  $A \cap B$ . What appears to be the relationship between these two sets? Explain.

---

In this section, we will learn how to prove certain relationships about sets. Two of the most basic types of relationships between sets are the equality relation and the subset relation. So if we are asked a question of the form, “How are the sets  $A$  and  $B$  related?”, we can answer the question if we can prove that the two sets are equal or that one set is a subset of the other set. There are other ways to answer this, but we will concentrate on these two for now. This is similar to asking a question about how two real numbers are related. Two real numbers can be related by the fact that they are equal or by the fact that one number is less than the other number.

### The Choose-an-Element Method

The method of proof we will use in this section can be called the **choose-an-element method**. This method was introduced in Preview Activity 1. This method is frequently used when we encounter a universal quantifier in a statement in the backward process. This statement often has the form

For each element with a given property, something happens.

Since most statements with a universal quantifier can be expressed in the form of a conditional statement, this statement could have the following equivalent form:

If an element has a given property, then something happens.

We will illustrate this with the proposition from Preview Activity 1. This proposition can be stated as follows:



*Let  $S$  be the set of all integers that are multiples of 6, and let  $T$  be the set of all even integers. Then  $S$  is a subset of  $T$ .*

In Preview Activity 1, we worked on a know-show table for this proposition. The key was that in the backward process, we encountered the following statement:

Each element of  $S$  is an element of  $T$  or, more precisely, if  $x \in S$ , then  $x \in T$ .

In this case, the “element” is an integer, the “given property” is that it is an element of  $S$ , and the “something that happens” is that the element is also an element of  $T$ . One way to approach this is to create a list of all elements with the given property and verify that for each one, the “something happens.” When the list is short, this may be a reasonable approach. However, as in this case, when the list is infinite (or even just plain long), this approach is not practical.

We overcome this difficulty by using the **choose-an-element method**, where we choose an arbitrary element with the given property. So in this case, we choose an integer  $x$  that is a multiple of 6. We cannot use a specific multiple of 6 (such as 12 or 24), but rather the only thing we can assume is that the integer satisfies the property that it is a multiple of 6. This is the key part of this method.

*Whenever we choose an arbitrary element with a given property, we are not selecting a specific element. Rather, the only thing we can assume about the element is the given property.*

It is important to realize that once we have chosen the arbitrary element, we have added information to the forward process. So in the know-show table for this proposition, we added the statement, “Let  $x \in S$ ” to the forward process. Following is a completed proof of this proposition following the outline of the know-show table from Preview Activity 1.

**Proposition 5.7.** *Let  $S$  be the set of all integers that are multiples of 6, and let  $T$  be the set of all even integers. Then  $S$  is a subset of  $T$ .*

**Proof.** Let  $S$  be the set of all integers that are multiples of 6, and let  $T$  be the set of all even integers. We will show that  $S$  is a subset of  $T$  by showing that if an integer  $x$  is an element of  $S$ , then it is also an element of  $T$ .

Let  $x \in S$ . (**Note:** The use of the word “let” is often an indication that the we are choosing an arbitrary element.) This means that  $x$  is a multiple of 6. Therefore,



there exists an integer  $m$  such that

$$x = 6m.$$

Since  $6 = 2 \cdot 3$ , this equation can be written in the form

$$x = 2(3m).$$

By closure properties of the integers,  $3m$  is an integer. Hence, this last equation proves that  $x$  must be even. Therefore, we have shown that if  $x$  is an element of  $S$ , then  $x$  is an element of  $T$ , and hence that  $S \subseteq T$ . ■

Having proved that  $S$  is a subset of  $T$ , we can now ask if  $S$  is actually equal to  $T$ . The work we did in Preview Activity 1 can help us answer this question. In that activity, we should have found several elements that are in  $T$  but not in  $S$ . For example, the integer 2 is in  $T$  since 2 is even but  $2 \notin S$  since 2 is not a multiple of 6. Therefore,  $S \neq T$  and we can also conclude that  $S$  is a proper subset of  $T$ .

One reason we do this in a “two-step” process is that it is much easier to work with the subset relation than the proper subset relation. The subset relation is defined by a conditional statement and most of our work in mathematics deals with proving conditional statements. In addition, the proper subset relation is a conjunction of two statements ( $S \subseteq T$  and  $S \neq T$ ) and so it is natural to deal with the two parts of the conjunction separately.

### Progress Check 5.8 (Subsets and Set Equality)

Let  $A = \{x \in \mathbb{Z} \mid x \text{ is a multiple of } 9\}$  and let  $B = \{x \in \mathbb{Z} \mid x \text{ is a multiple of } 3\}$ .

1. Is the set  $A$  a subset of  $B$ ? Justify your conclusion.
2. Is the set  $A$  equal to the set  $B$ ? Justify your conclusion.

### Progress Check 5.9 (Using the Choose-an-Element Method)

The Venn diagram in Preview Activity 2 suggests that the following proposition is true.

**Proposition 5.10.** *Let  $A$  and  $B$  be subsets of the universal set  $U$ . If  $A \subseteq B$ , then  $B^c \subseteq A^c$ .*

1. The conclusion of the conditional statement is  $B^c \subseteq A^c$ . Explain why we should try the choose-an-element method to prove this proposition.



2. Complete the following know-show table for this proposition and explain exactly where the choose-an-element method is used.

Step	Know	Reason
$P$	$A \subseteq B$	Hypothesis
$P1$	Let $x \in B^c$ .	Choose an arbitrary element of $B^c$ .
$P2$	If $x \in A$ , then $x \in B$ .	Definition of “subset”
$\vdots$	$\vdots$	$\vdots$
$Q1$	If $x \in B^c$ , then $x \in A^c$ .	
$Q$	$B^c \subseteq A^c$	Definition of “subset”
Step	Show	Reason

### Proving Set Equality

One way to prove that two sets are equal is to use Theorem 5.2 and prove each of the two sets is a subset of the other set. In particular, let  $A$  and  $B$  be subsets of some universal set. Theorem 5.2 states that  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ .

In Preview Activity 2, we created a Venn diagram that indicated that  $A - (A - B) = A \cap B$ . Following is a proof of this result. Notice where the choose-an-element method is used in each case.

**Proposition 5.11.** *Let  $A$  and  $B$  be subsets of some universal set. Then  $A - (A - B) = A \cap B$ .*

**Proof.** Let  $A$  and  $B$  be subsets of some universal set. We will prove that  $A - (A - B) = A \cap B$  by proving that  $A - (A - B) \subseteq A \cap B$  and that  $A \cap B \subseteq A - (A - B)$ .

First, let  $x \in A - (A - B)$ . This means that

$$x \in A \text{ and } x \notin (A - B).$$

We know that an element is in  $(A - B)$  if and only if it is in  $A$  and not in  $B$ . Since  $x \notin (A - B)$ , we conclude that  $x \notin A$  or  $x \in B$ . However, we also know that  $x \in A$  and so we conclude that  $x \in B$ . This proves that

$$x \in A \text{ and } x \in B.$$

This means that  $x \in A \cap B$ , and hence we have proved that  $A - (A - B) \subseteq A \cap B$ .



Now choose  $y \in A \cap B$ . This means that

$$y \in A \text{ and } y \in B.$$

We note that  $y \in (A - B)$  if and only if  $y \in A$  and  $y \notin B$  and hence,  $y \notin (A - B)$  if and only if  $y \notin A$  or  $y \in B$ . Since we have proved that  $y \in B$ , we conclude that  $y \notin (A - B)$ , and hence, we have established that  $y \in A$  and  $y \notin (A - B)$ . This proves that if  $y \in A \cap B$ , then  $y \in A - (A - B)$  and hence,  $A \cap B \subseteq A - (A - B)$ .

Since we have proved that  $A - (A - B) \subseteq A \cap B$  and  $A \cap B \subseteq A - (A - B)$ , we conclude that  $A - (A - B) = A \cap B$ . ■

### Progress Check 5.12 (Set Equality)

Prove the following proposition. To do so, prove each set is a subset of the other set by using the choose-an-element method.

**Proposition 5.13.** *Let  $A$  and  $B$  be subsets of some universal set. Then  $A - B = A \cap B^c$ .*

## Disjoint Sets

Earlier in this section, we discussed the concept of set equality and the relation of one set being a subset of another set. There are other possible relationships between two sets; one is that the sets are disjoint. Basically, two sets are disjoint if and only if they have nothing in common. We express this formally in the following definition.

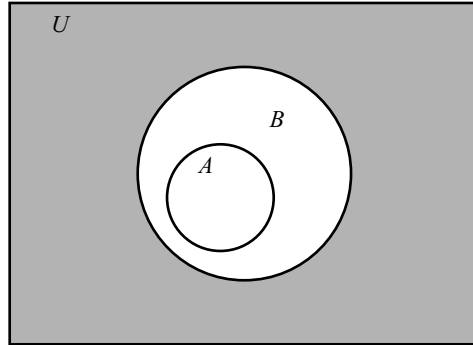
**Definition.** Let  $A$  and  $B$  be subsets of the universal set  $U$ . The sets  $A$  and  $B$  are said to be **disjoint** provided that  $A \cap B = \emptyset$ .

For example, the Venn diagram in Figure 5.5 shows two sets  $A$  and  $B$  with  $A \subseteq B$ . The shaded region is the region that represents  $B^c$ . From the Venn diagram, it appears that  $A \cap B^c = \emptyset$ . This means that  $A$  and  $B^c$  are disjoint. The preceding example suggests that the following proposition is true:

$$\text{If } A \subseteq B, \text{ then } A \cap B^c = \emptyset.$$

If we would like to prove this proposition, a reasonable “backward question” is, “How do we prove that a set (namely  $A \cap B^c$ ) is equal to the empty set?”



Figure 5.5: Venn Diagram with  $A \subseteq B$ 

This question seems difficult to answer since how do we prove that a set is empty? This is an instance where proving the contrapositive or using a proof by contradiction could be reasonable approaches. To illustrate these methods, let us assume the proposition we are trying to prove is of the following form:

If  $P$ , then  $T = \emptyset$ .

If we choose to prove the contrapositive or use a proof by contradiction, we will assume that  $T \neq \emptyset$ . These methods can be outlined as follows:

- The contrapositive of “If  $P$ , then  $T = \emptyset$ ” is, “If  $T \neq \emptyset$ , then  $\neg P$ .” So in this case, we would assume  $T \neq \emptyset$  and try to prove  $\neg P$ .
- Using a proof by contradiction, we would assume  $P$  and assume that  $T \neq \emptyset$ . From these two assumptions, we would attempt to derive a contradiction.

One advantage of these methods is that when we assume that  $T \neq \emptyset$ , then we know that there exists an element in the set  $T$ . We can then use that element in the rest of the proof. We will prove one of the conditional statements for Proposition 5.14 by proving its contrapositive. The proof of the other conditional statement associated with Proposition 5.14 is Exercise (10).

**Proposition 5.14.** *Let  $A$  and  $B$  be subsets of some universal set. Then  $A \subseteq B$  if and only if  $A \cap B^c = \emptyset$ .*

**Proof.** Let  $A$  and  $B$  be subsets of some universal set. We will first prove that if  $A \subseteq B$ , then  $A \cap B^c = \emptyset$ , by proving its contrapositive. That is, we will prove

If  $A \cap B^c \neq \emptyset$ , then  $A \not\subseteq B$ .

So assume that  $A \cap B^c \neq \emptyset$ . We will prove that  $A \not\subseteq B$  by proving that there must exist an element  $x$  such that  $x \in A$  and  $x \notin B$ .

Since  $A \cap B^c \neq \emptyset$ , there exists an element  $x$  that is in  $A \cap B^c$ . This means that

$$x \in A \text{ and } x \in B^c.$$

Now, the fact that  $x \in B^c$  means that  $x \notin B$ . Hence, we can conclude that

$$x \in A \text{ and } x \notin B.$$

This means that  $A \not\subseteq B$ , and hence, we have proved that if  $A \cap B^c \neq \emptyset$ , then  $A \not\subseteq B$ , and therefore, we have proved that if  $A \subseteq B$ , then  $A \cap B^c = \emptyset$ .

The proof that if  $A \cap B^c = \emptyset$ , then  $A \subseteq B$  is Exercise (10). ■

### Progress Check 5.15 (Proving Two Sets Are Disjoint)

It has been noted that it is often possible to prove that two sets are disjoint by using a proof by contradiction. In this case, we assume that the two sets are not disjoint and hence, their intersection is not empty. Use this method to prove that the following two sets are disjoint.

$$A = \{x \in \mathbb{Z} \mid x \equiv 3 \pmod{12}\} \quad \text{and} \quad B = \{y \in \mathbb{Z} \mid y \equiv 2 \pmod{8}\}.$$

### A Final Comment

We have used the choose-an-element method to prove Propositions 5.7, 5.11, and 5.14. Proofs involving sets that use this method are sometimes referred to as **element-chasing proofs**. This name is used since the basic method is to choose an arbitrary element from one set and “chase it” until you prove it must be in another set.

## Exercises for Section 5.2

\* 1. Let  $A = \{x \in \mathbb{R} \mid x^2 < 4\}$  and let  $B = \{x \in \mathbb{R} \mid x < 2\}$ .

(a) Is  $A \subseteq B$ ? Justify your conclusion with a proof or a counterexample.





(b) Is  $B \subseteq A$ ? Justify your conclusion with a proof or a counterexample.

2. Let  $A$ ,  $B$ , and  $C$  be subsets of a universal set  $U$ .

(a) Draw a Venn diagram with  $A \subseteq B$  and  $B \subseteq C$ . Does it appear that  $A \subseteq C$ ?

(b) Prove the following proposition:

If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

**Note:** This may seem like an obvious result. However, one of the reasons for this exercise is to provide practice at properly writing a proof that one set is a subset of another set. So we should start the proof by assuming that  $A \subseteq B$  and  $B \subseteq C$ . Then we should choose an arbitrary element of  $A$ .

\* 3. Let  $A = \{x \in \mathbb{Z} \mid x \equiv 7 \pmod{8}\}$  and  $B = \{x \in \mathbb{Z} \mid x \equiv 3 \pmod{4}\}$ .

(a) List at least five different elements of the set  $A$  and at least five elements of the set  $B$ .

(b) Is  $A \subseteq B$ ? Justify your conclusion with a proof or a counterexample.

(c) Is  $B \subseteq A$ ? Justify your conclusion with a proof or a counterexample.

4. Let  $C = \{x \in \mathbb{Z} \mid x \equiv 7 \pmod{9}\}$  and  $D = \{x \in \mathbb{Z} \mid x \equiv 1 \pmod{3}\}$ .

(a) List at least five different elements of the set  $C$  and at least five elements of the set  $D$ .

(b) Is  $C \subseteq D$ ? Justify your conclusion with a proof or a counterexample.

(c) Is  $D \subseteq C$ ? Justify your conclusion with a proof or a counterexample.

5. In each case, determine if  $A \subseteq B$ ,  $B \subseteq A$ ,  $A = B$ , or  $A \cap B = \emptyset$  or none of these.

\* (a)  $A = \{x \in \mathbb{Z} \mid x \equiv 2 \pmod{3}\}$  and  $B = \{y \in \mathbb{Z} \mid 6 \text{ divides } (2y - 4)\}$ .

(b)  $A = \{x \in \mathbb{Z} \mid x \equiv 3 \pmod{4}\}$  and  $B = \{y \in \mathbb{Z} \mid 3 \text{ divides } (y - 2)\}$ .

\* (c)  $A = \{x \in \mathbb{Z} \mid x \equiv 1 \pmod{5}\}$  and  $B = \{y \in \mathbb{Z} \mid y \equiv 7 \pmod{10}\}$ .

6. To prove the following set equalities, it may be necessary to use some of the properties of positive and negative real numbers. For example, it may be necessary to use the facts that:



- The product of two real numbers is positive if and only if the two real numbers are either both positive or both negative.
- The product of two real numbers is negative if and only if one of the two numbers is positive and the other is negative.

For example, if  $x(x - 2) < 0$ , then we can conclude that either (1)  $x < 0$  and  $x - 2 > 0$  or (2)  $x > 0$  and  $x - 2 < 0$ . However, in the first case, we must have  $x < 0$  and  $x > 2$ , and this is impossible. Therefore, we conclude that  $x > 0$  and  $x - 2 < 0$ , which means that  $0 < x < 2$ .

Use the choose-an-element method to prove each of the following:

- (a)  $\{x \in \mathbb{R} \mid x^2 - 3x - 10 < 0\} = \{x \in \mathbb{R} \mid -2 < x < 5\}$   
 (b)  $\{x \in \mathbb{R} \mid x^2 - 5x + 6 < 0\} = \{x \in \mathbb{R} \mid 2 < x < 3\}$   
 (c)  $\{x \in \mathbb{R} \mid x^2 \geq 4\} = \{x \in \mathbb{R} \mid x \leq -2\} \cup \{x \in \mathbb{R} \mid x \geq 2\}$

7. Let  $A$  and  $B$  be subsets of some universal set  $U$ . Prove each of the following:

- ★ (a)  $A \cap B \subseteq A$                       (c)  $A \cap A = A$                       ★ (e)  $A \cap \emptyset = \emptyset$   
 ★ (b)  $A \subseteq A \cup B$                       (d)  $A \cup A = A$                       (f)  $A \cup \emptyset = A$

8. Let  $A$  and  $B$  be subsets of some universal set  $U$ . From Proposition 5.10, we know that if  $A \subseteq B$ , then  $B^c \subseteq A^c$ . Now prove the following proposition:

For all sets  $A$  and  $B$  that are subsets of some universal set  $U$ ,  $A \subseteq B$  if and only if  $B^c \subseteq A^c$ .

9. Is the following proposition true or false? Justify your conclusion with a proof or a counterexample.

For all sets  $A$  and  $B$  that are subsets of some universal set  $U$ , the sets  $A \cap B$  and  $A - B$  are disjoint.

★ 10. Complete the proof of Proposition 5.14 by proving the following conditional statement:

Let  $A$  and  $B$  be subsets of some universal set. If  $A \cap B^c = \emptyset$ , then  $A \subseteq B$ .

11. Let  $A$ ,  $B$ ,  $C$ , and  $D$  be subsets of some universal set  $U$ . Are the following propositions true or false? Justify your conclusions.



- (a) If  $A \subseteq B$  and  $C \subseteq D$  and  $A$  and  $C$  are disjoint, then  $B$  and  $D$  are disjoint.
- (b) If  $A \subseteq B$  and  $C \subseteq D$  and  $B$  and  $D$  are disjoint, then  $A$  and  $C$  are disjoint.
12. Let  $A$ ,  $B$ , and  $C$  be subsets of a universal set  $U$ . Prove:
- \* (a) If  $A \subseteq B$ , then  $A \cap C \subseteq B \cap C$ .
- (b) If  $A \subseteq B$ , then  $A \cup C \subseteq B \cup C$ .
13. Let  $A$ ,  $B$ , and  $C$  be subsets of a universal set  $U$ . Are the following propositions true or false? Justify your conclusions.
- (a) If  $A \cap C \subseteq B \cap C$ , then  $A \subseteq B$ .
- (b) If  $A \cup C \subseteq B \cup C$ , then  $A \subseteq B$ .
- (c) If  $A \cup C = B \cup C$ , then  $A = B$ .
- (d) If  $A \cap C = B \cap C$ , then  $A = B$ .
- (e) If  $A \cup C = B \cup C$  and  $A \cap C = B \cap C$ , then  $A = B$ .
14. Prove the following proposition:
- For all sets  $A$ ,  $B$ , and  $C$  that are subsets of some universal set, if  $A \cap B = A \cap C$  and  $A^c \cap B = A^c \cap C$ , then  $B = C$ .
15. Are the following biconditional statements true or false? Justify your conclusion. If a biconditional statement is found to be false, you should clearly determine if one of the conditional statements within it is true and provide a proof of this conditional statement.
- \* (a) For all subsets  $A$  and  $B$  of some universal set  $U$ ,  $A \subseteq B$  if and only if  $A \cap B^c = \emptyset$ .
- \* (b) For all subsets  $A$  and  $B$  of some universal set  $U$ ,  $A \subseteq B$  if and only if  $A \cup B = B$ .
- (c) For all subsets  $A$  and  $B$  of some universal set  $U$ ,  $A \subseteq B$  if and only if  $A \cap B = A$ .
- (d) For all subsets  $A$ ,  $B$ , and  $C$  of some universal set  $U$ ,  $A \subseteq B \cup C$  if and only if  $A \subseteq B$  or  $A \subseteq C$ .
- (e) For all subsets  $A$ ,  $B$ , and  $C$  of some universal set  $U$ ,  $A \subseteq B \cap C$  if and only if  $A \subseteq B$  and  $A \subseteq C$ .
16. Let  $S$ ,  $T$ ,  $X$ , and  $Y$  be subsets of some universal set. Assume that

$$(i) S \cup T \subseteq X \cup Y; \quad (ii) S \cap T = \emptyset; \text{ and} \quad (iii) X \subseteq S.$$

- (a) Using assumption (i), what conclusion(s) can be made if it is known that  $a \in T$ ?
- (b) Using assumption (ii), what conclusion(s) can be made if it is known that  $a \in T$ ?
- (c) Using all three assumptions, either prove that  $T \subseteq Y$  or explain why it is not possible to do so.

### 17. Evaluation of Proofs

See the instructions for Exercise (19) on page 100 from Section 3.1.

- (a) Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set. If  $A \not\subseteq B$  and  $B \not\subseteq C$ , then  $A \not\subseteq C$ .

**Proof.** We assume that  $A$ ,  $B$ , and  $C$  are subsets of some universal set and that  $A \not\subseteq B$  and  $B \not\subseteq C$ . This means that there exists an element  $x$  in  $A$  that is not in  $B$  and there exists an element  $x$  that is in  $B$  and not in  $C$ . Therefore,  $x \in A$  and  $x \notin C$ , and we have proved that  $A \not\subseteq C$ . ■

- (b) Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set. If  $A \cap B = A \cap C$ , then  $B = C$ .

**Proof.** We assume that  $A \cap B = A \cap C$  and will prove that  $B = C$ . We will first prove that  $B \subseteq C$ .

So let  $x \in B$ . If  $x \in A$ , then  $x \in A \cap B$ , and hence,  $x \in A \cap C$ . From this we can conclude that  $x \in C$ . If  $x \notin A$ , then  $x \notin A \cap B$ , and hence,  $x \notin A \cap C$ . However, since  $x \in B$ , we may conclude that  $x \in C$ . Therefore,  $B \subseteq C$ .

The proof that  $C \subseteq B$  may be done in a similar manner. Hence,  $B = C$ . ■

- (c) Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set. If  $A \not\subseteq B$  and  $B \subseteq C$ , then  $A \not\subseteq C$ .

**Proof.** Assume that  $A \not\subseteq B$  and  $B \subseteq C$ . Since  $A \not\subseteq B$ , there exists an element  $x$  such that  $x \in A$  and  $x \notin B$ . Since  $B \subseteq C$ , we may conclude that  $x \notin C$ . Hence,  $x \in A$  and  $x \notin C$ , and we have proved that  $A \not\subseteq C$ . ■

## Explorations and Activities

**18. Using the Choose-an-Element Method in a Different Setting.** We have used the choose-an-element method to prove results about sets. This method, however, is a general proof technique and can be used in settings other than set theory. It is often used whenever we encounter a universal quantifier in a statement in the backward process. Consider the following proposition.

**Proposition 5.16.** *Let  $a$ ,  $b$ , and  $t$  be integers with  $t \neq 0$ . If  $t$  divides  $a$  and  $t$  divides  $b$ , then for all integers  $x$  and  $y$ ,  $t$  divides  $(ax + by)$ .*

(a) Whenever we encounter a new proposition, it is a good idea to explore the proposition by looking at specific examples. For example, let  $a = 20$ ,  $b = 12$ , and  $t = 4$ . In this case,  $t \mid a$  and  $t \mid b$ . In each of the following cases, determine the value of  $(ax + by)$  and determine if  $t$  divides  $(ax + by)$ .

- |                     |                      |
|---------------------|----------------------|
| i. $x = 1, y = 1$   | iv. $x = 2, y = -3$  |
| ii. $x = 1, y = -1$ | v. $x = -2, y = 3$   |
| iii. $x = 2, y = 2$ | vi. $x = -2, y = -5$ |

(b) Repeat Part (18a) with  $a = 21$ ,  $b = -6$ , and  $t = 3$ .

Notice that the conclusion of the conditional statement in this proposition involves the universal quantifier. So in the backward process, we would have

$Q$ : For all integers  $x$  and  $y$ ,  $t$  divides  $ax + by$ .

The “elements” in this sentence are the integers  $x$  and  $y$ . In this case, these integers have no “given property” other than that they are integers. The “something that happens” is that  $t$  divides  $(ax + by)$ . This means that in the forward process, we can use the hypothesis of the proposition and choose integers  $x$  and  $y$ . That is, in the forward process, we could have

$P$ :  $a$ ,  $b$ , and  $t$  are integers with  $t \neq 0$ ,  $t$  divides  $a$  and  $t$  divides  $b$ .

$P1$ : Let  $x \in \mathbb{Z}$  and let  $y \in \mathbb{Z}$ .

(c) Complete the following proof of Proposition 5.16.



**Proof.** Let  $a$ ,  $b$ , and  $t$  be integers with  $t \neq 0$ , and assume that  $t$  divides  $a$  and  $t$  divides  $b$ . We will prove that for all integers  $x$  and  $y$ ,  $t$  divides  $(ax + by)$ .

So let  $x \in \mathbb{Z}$  and let  $y \in \mathbb{Z}$ . Since  $t$  divides  $a$ , there exists an integer  $m$  such that . . . . ■

### 5.3 Properties of Set Operations

#### Preview Activity 1 (Exploring a Relationship between Two Sets)

Let  $A$  and  $B$  be subsets of some universal set  $U$ .

1. Draw two general Venn diagrams for the sets  $A$  and  $B$ . On one, shade the region that represents  $(A \cup B)^c$ , and on the other, shade the region that represents  $A^c \cap B^c$ . Explain carefully how you determined these regions.
2. Based on the Venn diagrams in Part (1), what appears to be the relationship between the sets  $(A \cup B)^c$  and  $A^c \cap B^c$ ?

Some of the properties of set operations are closely related to some of the logical operators we studied in Section 2.1. This is due to the fact that set intersection is defined using a conjunction (and), and set union is defined using a disjunction (or). For example, if  $A$  and  $B$  are subsets of some universal set  $U$ , then an element  $x$  is in  $A \cup B$  if and only if  $x \in A$  or  $x \in B$ .

3. Use one of De Morgan's Laws (Theorem 2.8 on page 48) to explain carefully what it means to say that an element  $x$  is not in  $A \cup B$ .
4. What does it mean to say that an element  $x$  is in  $A^c$ ? What does it mean to say that an element  $x$  is in  $B^c$ ?
5. Explain carefully what it means to say that an element  $x$  is in  $A^c \cap B^c$ .
6. Compare your response in Part (3) to your response in Part (5). Are they equivalent? Explain.
7. How do you think the sets  $(A \cup B)^c$  and  $A^c \cap B^c$  are related? Is this consistent with the Venn diagrams from Part (1)?

**Preview Activity 2 (Proving that Statements Are Equivalent)**

1. Let  $X$ ,  $Y$ , and  $Z$  be statements. Complete a truth table for  $[(X \rightarrow Y) \wedge (Y \rightarrow Z)] \rightarrow (X \rightarrow Z)$ .
2. Assume that  $P$ ,  $Q$ , and  $R$  are statements and that we have proven that the following conditional statements are true:
  - If  $P$  then  $Q$  ( $P \rightarrow Q$ ).
  - If  $R$  then  $P$  ( $R \rightarrow P$ ).
  - If  $Q$  then  $R$  ( $Q \rightarrow R$ ).

Explain why each of the following statements is true.

- (a)  $P$  if and only if  $Q$  ( $P \leftrightarrow Q$ ).      (c)  $R$  if and only if  $P$  ( $R \leftrightarrow P$ ).  
 (b)  $Q$  if and only if  $R$  ( $Q \leftrightarrow R$ ).

Remember that  $X \leftrightarrow Y$  is logically equivalent to  $(X \rightarrow Y) \wedge (Y \rightarrow X)$ .

**Algebra of Sets – Part 1**

This section contains many results concerning the properties of the set operations. We have already proved some of the results. Others will be proved in this section or in the exercises. The primary purpose of this section is to have in one place many of the properties of set operations that we may use in later proofs. These results are part of what is known as the **algebra of sets** or as **set theory**.

**Theorem 5.17.** *Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ . Then*

- $A \cap B \subseteq A$  and  $A \subseteq A \cup B$ .
- If  $A \subseteq B$ , then  $A \cap C \subseteq B \cap C$  and  $A \cup C \subseteq B \cup C$ .

**Proof.** The first part of this theorem was included in Exercise (7) from Section 5.2. The second part of the theorem was Exercise (12) from Section 5.2. ■

The next theorem provides many of the properties of set operations dealing with intersection and union. Many of these results may be intuitively obvious, but to be complete in the development of set theory, we should prove all of them. We choose to prove only some of them and leave some as exercises.



**Theorem 5.18 (Algebra of Set Operations).** *Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ . Then all of the following equalities hold.*

<i>Properties of the Empty Set and the Universal Set</i>	$A \cap \emptyset = \emptyset$	$A \cap U = A$
	$A \cup \emptyset = A$	$A \cup U = U$
<i>Idempotent Laws</i>	$A \cap A = A$	$A \cup A = A$
<i>Commutative Laws</i>	$A \cap B = B \cap A$	$A \cup B = B \cup A$
<i>Associative Laws</i>	$(A \cap B) \cap C = A \cap (B \cap C)$	
	$(A \cup B) \cup C = A \cup (B \cup C)$	
<i>Distributive Laws</i>	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	

Before proving some of these properties, we note that in Section 5.2, we learned that we can prove that two sets are equal by proving that each one is a subset of the other one. However, we also know that if  $S$  and  $T$  are both subsets of a universal set  $U$ , then

$$S = T \text{ if and only if for each } x \in U, x \in S \text{ if and only if } x \in T.$$

We can use this to prove that two sets are equal by choosing an element from one set and chasing the element to the other set through a sequence of “if and only if” statements. We now use this idea to prove one of the commutative laws.

### Proof of One of the Commutative Laws in Theorem 5.18

**Proof.** We will prove that  $A \cap B = B \cap A$ . Let  $x \in U$ . Then

$$x \in A \cap B \text{ if and only if } x \in A \text{ and } x \in B. \quad (1)$$

However, we know that if  $P$  and  $Q$  are statements, then  $P \wedge Q$  is logically equivalent to  $Q \wedge P$ . Consequently, we can conclude that

$$x \in A \text{ and } x \in B \text{ if and only if } x \in B \text{ and } x \in A. \quad (2)$$

Now we know that

$$x \in B \text{ and } x \in A \text{ if and only if } x \in B \cap A. \quad (3)$$

This means that we can use (1), (2), and (3) to conclude that

$$x \in A \cap B \text{ if and only if } x \in B \cap A,$$

and, hence, we have proved that  $A \cap B = B \cap A$ . ■





**Progress Check 5.19 (Exploring a Distributive Property)**

We can use Venn diagrams to explore the more complicated properties in Theorem 5.18, such as the associative and distributive laws. To that end, let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ .

1. Draw two general Venn diagrams for the sets  $A$ ,  $B$ , and  $C$ . On one, shade the region that represents  $A \cup (B \cap C)$ , and on the other, shade the region that represents  $(A \cup B) \cap (A \cup C)$ . Explain carefully how you determined these regions.
2. Based on the Venn diagrams in Part (1), what appears to be the relationship between the sets  $A \cup (B \cap C)$  and  $(A \cup B) \cap (A \cup C)$ ?

**Proof of One of the Distributive Laws in Theorem 5.18**

We will now prove the distributive law explored in Progress Check 5.19. Notice that we will prove two subset relations, and that for each subset relation, we will begin by choosing an arbitrary element from a set. Also notice how nicely a proof dealing with the union of two sets can be broken into cases.

**Proof.** Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ . We will prove that  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  by proving that each set is a subset of the other set.

We will first prove that  $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ . We let  $x \in A \cup (B \cap C)$ . Then  $x \in A$  or  $x \in B \cap C$ .

So in one case, if  $x \in A$ , then  $x \in A \cup B$  and  $x \in A \cup C$ . This means that  $x \in (A \cup B) \cap (A \cup C)$ .

On the other hand, if  $x \in B \cap C$ , then  $x \in B$  and  $x \in C$ . But  $x \in B$  implies that  $x \in A \cup B$ , and  $x \in C$  implies that  $x \in A \cup C$ . Since  $x$  is in both sets, we conclude that  $x \in (A \cup B) \cap (A \cup C)$ . So in both cases, we see that  $x \in (A \cup B) \cap (A \cup C)$ , and this proves that  $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ .

We next prove that  $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ . So let  $y \in (A \cup B) \cap (A \cup C)$ . Then,  $y \in A \cup B$  and  $y \in A \cup C$ . We must prove that  $y \in A \cup (B \cap C)$ . We will consider the two cases where  $y \in A$  or  $y \notin A$ . In the case where  $y \in A$ , we see that  $y \in A \cup (B \cap C)$ .

So we consider the case that  $y \notin A$ . It has been established that  $y \in A \cup B$  and  $y \in A \cup C$ . Since  $y \notin A$  and  $y \in A \cup B$ ,  $y$  must be an element of  $B$ . Similarly,



since  $y \notin A$  and  $y \in A \cup C$ ,  $y$  must be an element of  $C$ . Thus,  $y \in B \cap C$  and, hence,  $y \in A \cup (B \cap C)$ .

In both cases, we have proved that  $y \in A \cup (B \cap C)$ . This proves that  $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ . The two subset relations establish the equality of the two sets. Thus,  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ . ■

### Important Properties of Set Complements

The three main set operations are union, intersection, and complementation. Theorems 5.18 and 5.17 deal with properties of unions and intersections. The next theorem states some basic properties of complements and the important relations dealing with complements of unions and complements of intersections. Two relationships in the next theorem are known as **De Morgan's Laws** for sets and are closely related to De Morgan's Laws for statements.

**Theorem 5.20.** *Let  $A$  and  $B$  be subsets of some universal set  $U$ . Then the following are true:*

<i>Basic Properties</i>	$(A^c)^c = A$ $A - B = A \cap B^c$
<i>Empty Set and Universal Set</i>	$A - \emptyset = A$ and $A - U = \emptyset$ $\emptyset^c = U$ and $U^c = \emptyset$
<i>De Morgan's Laws</i>	$(A \cap B)^c = A^c \cup B^c$ $(A \cup B)^c = A^c \cap B^c$
<i>Subsets and Complements</i>	$A \subseteq B$ if and only if $B^c \subseteq A^c$

**Proof.** We will only prove one of De Morgan's Laws, namely, the one that was explored in Preview Activity 1. The proofs of the other parts are left as exercises. Let  $A$  and  $B$  be subsets of some universal set  $U$ . We will prove that  $(A \cup B)^c = A^c \cap B^c$  by proving that an element is in  $(A \cup B)^c$  if and only if it is in  $A^c \cap B^c$ . So let  $x$  be in the universal set  $U$ . Then

$$x \in (A \cup B)^c \text{ if and only if } x \notin A \cup B, \quad (1)$$

and

$$x \notin A \cup B \text{ if and only if } x \notin A \text{ and } x \notin B. \quad (2)$$

Combining (1) and (2), we see that

$$x \in (A \cup B)^c \text{ if and only if } x \notin A \text{ and } x \notin B. \quad (3)$$



In addition, we know that

$$x \notin A \text{ and } x \notin B \text{ if and only if } x \in A^c \text{ and } x \in B^c, \quad (4)$$

and this is true if and only if  $x \in A^c \cap B^c$ . So we can use (3) and (4) to conclude that

$$x \in (A \cup B)^c \text{ if and only if } x \in A^c \cap B^c,$$

and, hence, that  $(A \cup B)^c = A^c \cap B^c$ . ■

### Progress Check 5.21 (Using the Algebra of Sets)

1. Draw two general Venn diagrams for the sets  $A$ ,  $B$ , and  $C$ . On one, shade the region that represents  $(A \cup B) - C$ , and on the other, shade the region that represents  $(A - C) \cup (B - C)$ . Explain carefully how you determined these regions and why they indicate that  $(A \cup B) - C = (A - C) \cup (B - C)$ .

It is possible to prove the relationship suggested in Part (1) by proving that each set is a subset of the other set. However, the results in Theorems 5.18 and 5.20 can be used to prove other results about set operations. When we do this, we say that we are using the algebra of sets to prove the result. For example, we can start by using one of the basic properties in Theorem 5.20 to write

$$(A \cup B) - C = (A \cup B) \cap C^c.$$

We can then use one of the commutative properties to write

$$\begin{aligned} (A \cup B) - C &= (A \cup B) \cap C^c \\ &= C^c \cap (A \cup B). \end{aligned}$$

2. Determine which properties from Theorems 5.18 and 5.20 justify each of the last three steps in the following outline of the proof that  $(A \cup B) - C = (A - C) \cup (B - C)$ .

$$\begin{aligned} (A \cup B) - C &= (A \cup B) \cap C^c && \text{(Theorem 5.20)} \\ &= C^c \cap (A \cup B) && \text{(Commutative Property)} \\ &= (C^c \cap A) \cup (C^c \cap B) \\ &= (A \cap C^c) \cup (B \cap C^c) \\ &= (A - C) \cup (B - C) \end{aligned}$$



**Note:** It is sometimes difficult to use the properties in the theorems when the theorems use the same letters to represent the sets as those being used in the current problem. For example, one of the distributive properties from Theorems 5.18 can be written as follows: For all sets  $X$ ,  $Y$ , and  $Z$  that are subsets of a universal set  $U$ ,

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z).$$

### Proving that Statements Are Equivalent

When we have a list of three statements  $P$ ,  $Q$ , and  $R$  such that each statement in the list is equivalent to the other two statements in the list, we say that the three statements are equivalent. This means that each of the statements in the list implies each of the other statements in the list.

The purpose of Preview Activity 2 was to provide one way to prove that three (or more) statements are equivalent. The basic idea is to prove a sequence of conditional statements so that there is an unbroken chain of conditional statements from each statement to every other statement. This method of proof will be used in Theorem 5.22.

**Theorem 5.22.** *Let  $A$  and  $B$  be subsets of some universal set  $U$ . The following are equivalent:*

1.  $A \subseteq B$
2.  $A \cap B^c = \emptyset$
3.  $A^c \cup B = U$

**Proof.** To prove that these are equivalent conditions, we will prove that (1) implies (2), that (2) implies (3), and that (3) implies (1).

Let  $A$  and  $B$  be subsets of some universal set  $U$ . We have proved that (1) implies (2) in Proposition 5.14.

To prove that (2) implies (3), we will assume that  $A \cap B^c = \emptyset$  and use the fact that  $\emptyset^c = U$ . We then see that

$$(A \cap B^c)^c = \emptyset^c.$$

Then, using one of De Morgan's Laws, we obtain

$$\begin{aligned} A^c \cup (B^c)^c &= U \\ A^c \cup B &= U. \end{aligned}$$



This completes the proof that (2) implies (3).

We now need to prove that (3) implies (1). We assume that  $A^c \cup B = U$  and will prove that  $A \subseteq B$  by proving that every element of  $A$  must be in  $B$ .

So let  $x \in A$ . Then we know that  $x \notin A^c$ . However,  $x \in U$  and since  $A^c \cup B = U$ , we can conclude that  $x \in A^c \cup B$ . Since  $x \notin A^c$ , we conclude that  $x \in B$ . This proves that  $A \subseteq B$  and hence that (3) implies (1).

Since we have now proved that (1) implies (2), that (2) implies (3), and that (3) implies (1), we have proved that the three conditions are equivalent. ■

## Exercises for Section 5.3

1. Let  $A$  be a subset of some universal set  $U$ . Prove each of the following (from Theorem 5.20):

★ (a)  $(A^c)^c = A$

★ (c)  $\emptyset^c = U$

(b)  $A - \emptyset = A$

(d)  $U^c = \emptyset$

- ★ 2. Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ . As part of Theorem 5.18, we proved one of the distributive laws. Prove the other one. That is, prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

3. Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ . As part of Theorem 5.20, we proved one of De Morgan's Laws. Prove the other one. That is, prove that

$$(A \cap B)^c = A^c \cup B^c.$$

4. Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ .

- ★ (a) Draw two general Venn diagrams for the sets  $A$ ,  $B$ , and  $C$ . On one, shade the region that represents  $A - (B \cup C)$ , and on the other, shade the region that represents  $(A - B) \cap (A - C)$ . Based on the Venn diagrams, make a conjecture about the relationship between the sets  $A - (B \cup C)$  and  $(A - B) \cap (A - C)$ .
- (b) Use the choose-an-element method to prove the conjecture from Exercise (4a).



\* (c) Use the algebra of sets to prove the conjecture from Exercise (4a).

5. Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ .

(a) Draw two general Venn diagrams for the sets  $A$ ,  $B$ , and  $C$ . On one, shade the region that represents  $A - (B \cap C)$ , and on the other, shade the region that represents  $(A - B) \cup (A - C)$ . Based on the Venn diagrams, make a conjecture about the relationship between the sets  $A - (B \cap C)$  and  $(A - B) \cup (A - C)$ .

(b) Use the choose-an-element method to prove the conjecture from Exercise (5a).

(c) Use the algebra of sets to prove the conjecture from Exercise (5a).

6. Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ . Prove or disprove each of the following:

\* (a)  $(A \cap B) - C = (A - C) \cap (B - C)$

(b)  $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

7. Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ .

(a) Draw two general Venn diagrams for the sets  $A$ ,  $B$ , and  $C$ . On one, shade the region that represents  $A - (B - C)$ , and on the other, shade the region that represents  $(A - B) - C$ . Based on the Venn diagrams, make a conjecture about the relationship between the sets  $A - (B - C)$  and  $(A - B) - C$ . (Are the two sets equal? If not, is one of the sets a subset of the other set?)

(b) Prove the conjecture from Exercise (7a).

8. Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ .

(a) Draw two general Venn diagrams for the sets  $A$ ,  $B$ , and  $C$ . On one, shade the region that represents  $A - (B - C)$ , and on the other, shade the region that represents  $(A - B) \cup (A - C^c)$ . Based on the Venn diagrams, make a conjecture about the relationship between the sets  $A - (B - C)$  and  $(A - B) \cup (A - C^c)$ . (Are the two sets equal? If not, is one of the sets a subset of the other set?)

(b) Prove the conjecture from Exercise (8a).

9. Let  $A$  and  $B$  be subsets of some universal set  $U$ .

\* (a) Prove that  $A$  and  $B - A$  are disjoint sets.



(b) Prove that  $A \cup B = A \cup (B - A)$ .

10. Let  $A$  and  $B$  be subsets of some universal set  $U$ .

(a) Prove that  $A - B$  and  $A \cap B$  are disjoint sets.

(b) Prove that  $A = (A - B) \cup (A \cap B)$ .

11. Let  $A$  and  $B$  be subsets of some universal set  $U$ . Prove or disprove each of the following:

(a)  $A - (A \cap B^c) = A \cap B$

(b)  $(A^c \cup B)^c \cap A = A - B$

(c)  $(A \cup B) - A = B - A$

(d)  $(A \cup B) - B = A - (A \cap B)$

## 12. Evaluation of proofs

See the instructions for Exercise (19) on page 100 from Section 3.1.

(a) If  $A$ ,  $B$ , and  $C$  are subsets of some universal set  $U$ , then  $A - (B - C) = A - (B \cup C)$ .

*Proof.*

$$\begin{aligned} A - (B - C) &= (A - B) - (A - C) \\ &= (A \cap B^c) \cap (A \cap C^c) \\ &= A \cap (B^c \cap C^c) \\ &= A \cap (B \cup C)^c \\ &= A - (B \cup C) \end{aligned}$$

■

(b) If  $A$ ,  $B$ , and  $C$  are subsets of some universal set  $U$ , then  $A - (B \cup C) = (A - B) \cap (A - C)$ .

*Proof.* We first write  $A - (B \cup C) = A \cap (B \cup C)^c$  and then use one of De Morgan's Laws to obtain

$$A - (B \cup C) = A \cap (B^c \cap C^c).$$

We now use the fact that  $A = A \cap A$  and obtain

$$\begin{aligned} A - (B \cup C) &= A \cap A \cap B^c \cap C^c = (A \cap B^c) \cap (A \cap C^c) = \\ &= (A - B) \cap (A - C). \end{aligned}$$

■



## Explorations and Activities

- 13. (Comparison to Properties of the Real Numbers).** The following are some of the basic properties of addition and multiplication of real numbers.

**Commutative Laws:**  $a + b = b + a$ , for all  $a, b \in \mathbb{R}$ .  
 $a \cdot b = b \cdot a$ , for all  $a, b \in \mathbb{R}$ .

**Associative Laws:**  $(a + b) + c = a + (b + c)$ , for all  $a, b, c \in \mathbb{R}$ .  
 $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ , for all  $a, b, c \in \mathbb{R}$ .

**Distributive Law:**  $a \cdot (b + c) = a \cdot b + a \cdot c$ , for all  $a, b, c \in \mathbb{R}$ .

**Additive Identity:** For all  $a \in \mathbb{R}$ ,  $a + 0 = a = 0 + a$ .

**Multiplicative Identity:** For all  $a \in \mathbb{R}$ ,  $a \cdot 1 = a = 1 \cdot a$ .

**Additive Inverses:** For all  $a \in \mathbb{R}$ ,  $a + (-a) = 0 = (-a) + a$ .

**Multiplicative Inverses:** For all  $a \in \mathbb{R}$  with  $a \neq 0$ ,  $a \cdot a^{-1} = 1 = a^{-1} \cdot a$ .

Discuss the similarities and differences among the properties of addition and multiplication of real numbers and the properties of union and intersection of sets.

## 5.4 Cartesian Products

### Preview Activity 1 (An Equation with Two Variables)

In Section 2.3, we introduced the concept of the **truth set of an open sentence with one variable**. This was defined to be the set of all elements in the universal set that can be substituted for the variable to make the open sentence a true statement.

In previous mathematics courses, we have also had experience with open sentences with two variables. For example, if we assume that  $x$  and  $y$  represent real numbers, then the equation

$$2x + 3y = 12$$





is an open sentence with two variables. An element of the truth set of this open sentence (also called a solution of the equation) is an ordered pair  $(a, b)$  of real numbers so that when  $a$  is substituted for  $x$  and  $b$  is substituted for  $y$ , the open sentence becomes a true statement (a true equation in this case). For example, we see that the ordered pair  $(6, 0)$  is in the truth set for this open sentence since

$$2 \cdot 6 + 3 \cdot 0 = 12$$

is a true statement. On the other hand, the ordered pair  $(4, 1)$  is not in the truth set for this open sentence since

$$2 \cdot 4 + 3 \cdot 1 = 12$$

is a false statement.

**Important Note:** The order of the two numbers in the ordered pair is very important. We are using the convention that the first number is to be substituted for  $x$  and the second number is to be substituted for  $y$ . With this convention,  $(3, 2)$  is a solution of the equation  $2x + 3y = 12$ , but  $(2, 3)$  is not a solution of this equation.

1. List six different elements of the truth set (often called the solution set) of the open sentence with two variables  $2x + 3y = 12$ .
2. From previous mathematics courses, we know that the graph of the equation  $2x + 3y = 12$  is a straight line. Sketch the graph of the equation  $2x + 3y = 12$  in the  $xy$ -coordinate plane. What does the graph of the equation  $2x + 3y = 12$  show?
3. Write a description of the solution set  $S$  of the equation  $2x + 3y = 12$  using set builder notation.

---

### Preview Activity 2 (The Cartesian Product of Two Sets)

In Preview Activity 1, we worked with ordered pairs without providing a formal definition of an ordered pair. We instead relied on your previous work with ordered pairs, primarily from graphing equations with two variables. Following is a formal definition of an ordered pair.



**Definition.** Let  $A$  and  $B$  be sets. An **ordered pair** (with first element from  $A$  and second element from  $B$ ) is a single pair of objects, denoted by  $(a, b)$ , with  $a \in A$  and  $b \in B$  and an implied order. This means that for two ordered pairs to be equal, they must contain exactly the same objects in the same order. That is, if  $a, c \in A$  and  $b, d \in B$ , then

$$(a, b) = (c, d) \text{ if and only if } a = c \text{ and } b = d.$$

The objects in the ordered pair are called the **coordinates** of the ordered pair. In the ordered pair  $(a, b)$ ,  $a$  is the **first coordinate** and  $b$  is the **second coordinate**.

We will now introduce a new set operation that gives a way of combining elements from two given sets to form ordered pairs. The basic idea is that we will create a set of ordered pairs.

**Definition.** If  $A$  and  $B$  are sets, then the **Cartesian product**,  $A \times B$ , of  $A$  and  $B$  is the set of all ordered pairs  $(x, y)$  where  $x \in A$  and  $y \in B$ . We use the notation  $A \times B$  for the Cartesian product of  $A$  and  $B$ , and using set builder notation, we can write

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

We frequently read  $A \times B$  as “ $A$  cross  $B$ .” In the case where the two sets are the same, we will write  $A^2$  for  $A \times A$ . That is,

$$A^2 = A \times A = \{(a, b) \mid a \in A \text{ and } b \in A\}.$$

Let  $A = \{1, 2, 3\}$  and  $B = \{a, b\}$ .

1. Is the ordered pair  $(3, a)$  in the Cartesian product  $A \times B$ ? Explain.
2. Is the ordered pair  $(3, a)$  in the Cartesian product  $A \times A$ ? Explain.
3. Is the ordered pair  $(3, 1)$  in the Cartesian product  $A \times A$ ? Explain.
4. Use the roster method to specify all the elements of  $A \times B$ . (Remember that the elements of  $A \times B$  will be ordered pairs.)
5. Use the roster method to specify all of the elements of the set  $A \times A = A^2$ .



6. For any sets  $C$  and  $D$ , explain carefully what it means to say that the ordered pair  $(x, y)$  is not in the Cartesian product  $C \times D$ .

## Cartesian Products

When working with Cartesian products, it is important to remember that the Cartesian product of two sets is itself a set. As a set, it consists of a collection of elements. In this case, the elements of a Cartesian product are ordered pairs. We should think of an ordered pair as a single object that consists of two other objects in a specified order. For example,

- If  $a \neq 1$ , then the ordered pair  $(1, a)$  is not equal to the ordered pair  $(a, 1)$ . That is,  $(1, a) \neq (a, 1)$ .
- If  $A = \{1, 2, 3\}$  and  $B = \{a, b\}$ , then the ordered pair  $(3, a)$  is an element of the set  $A \times B$ . That is,  $(3, a) \in A \times B$ .
- If  $A = \{1, 2, 3\}$  and  $B = \{a, b\}$ , then the ordered pair  $(5, a)$  is not an element of the set  $A \times B$  since  $5 \notin A$ . That is,  $(5, a) \notin A \times B$ .

In Section 5.3, we studied certain properties of set union, set intersection, and set complements, which we called the algebra of sets. We will now begin something similar for Cartesian products. We begin by examining some specific examples in Progress Check 5.23 and a little later in Progress Check 5.24.

### Progress Check 5.23 (Relationships between Cartesian Products)

Let  $A = \{1, 2, 3\}$ ,  $T = \{1, 2\}$ ,  $B = \{a, b\}$ , and  $C = \{a, c\}$ . We can then form new sets from all of the set operations we have studied. For example,  $B \cap C = \{a\}$ , and so

$$A \times (B \cap C) = \{(1, a), (2, a), (3, a)\}.$$

1. Use the roster method to list all of the elements (ordered pairs) in each of the following sets:
 

(a) $A \times B$	(f) $A \times (B \cup C)$
(b) $T \times B$	(g) $(A \times B) \cup (A \times C)$
(c) $A \times C$	(h) $A \times (B - C)$
(d) $A \times (B \cap C)$	(i) $(A \times B) - (A \times C)$
(e) $(A \times B) \cap (A \times C)$	(j) $B \times A$



2. List all the relationships between the sets in Part (1) that you observe.

## The Cartesian Plane

In Preview Activity 1, we sketched the graph of the equation  $2x + 3y = 12$  in the  $xy$ -plane. This  $xy$ -plane, with which you are familiar, is a representation of the set  $\mathbb{R} \times \mathbb{R}$  or  $\mathbb{R}^2$ . This plane is called the **Cartesian plane**.

The basic idea is that each ordered pair of real numbers corresponds to a point in the plane, and each point in the plane corresponds to an ordered pair of real numbers. This geometric representation of  $\mathbb{R}^2$  is an extension of the geometric representation of  $\mathbb{R}$  as a straight line whose points correspond to real numbers.

Since the Cartesian product  $\mathbb{R}^2$  corresponds to the Cartesian plane, the Cartesian product of two subsets of  $\mathbb{R}$  corresponds to a subset of the Cartesian plane. For example, if  $A$  is the interval  $[1, 3]$ , and  $B$  is the interval  $[2, 5]$ , then

$$A \times B = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 3 \text{ and } 2 \leq y \leq 5\}.$$

A graph of the set  $A \times B$  can then be drawn in the Cartesian plane as shown in Figure 5.6.

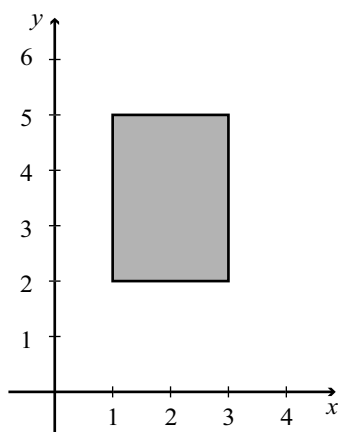


Figure 5.6: Cartesian Product  $A \times B$

This illustrates that the graph of a Cartesian product of two intervals of finite length in  $\mathbb{R}$  corresponds to the interior of a rectangle and possibly some or all of its

boundary. The solid line for the boundary in Figure 5.6 indicates that the boundary is included. In this case, the Cartesian product contained all of the boundary of the rectangle. When the graph does not contain a portion of the boundary, we usually draw that portion of the boundary with a dotted line.

**Note: A Caution about Notation.** The standard notation for an open interval in  $\mathbb{R}$  is the same as the notation for an ordered pair, which is an element of  $\mathbb{R} \times \mathbb{R}$ . We need to use the context in which the notation is used to determine which interpretation is intended. For example,

- If we write  $(\sqrt{2}, 7) \in \mathbb{R} \times \mathbb{R}$ , then we are using  $(\sqrt{2}, 7)$  to represent an ordered pair of real numbers.
- If we write  $(1, 2) \times \{4\}$ , then we are interpreting  $(1, 2)$  as an open interval. We could write

$$(1, 2) \times \{4\} = \{(x, 4) \mid 1 < x < 2\}.$$

The following progress check explores some of the same ideas explored in Progress Check 5.23 except that intervals of real numbers are used for the sets.

#### Progress Check 5.24 (Cartesian Products of Intervals)

We will use the following intervals that are subsets of  $\mathbb{R}$ .

$$A = [0, 2] \quad T = (1, 2) \quad B = [2, 4) \quad C = (3, 5]$$

1. Draw a graph of each of the following subsets of the Cartesian plane and write each subset using set builder notation.

- |                                      |                                      |
|--------------------------------------|--------------------------------------|
| (a) $A \times B$                     | (f) $A \times (B \cup C)$            |
| (b) $T \times B$                     | (g) $(A \times B) \cup (A \times C)$ |
| (c) $A \times C$                     | (h) $A \times (B - C)$               |
| (d) $A \times (B \cap C)$            | (i) $(A \times B) - (A \times C)$    |
| (e) $(A \times B) \cap (A \times C)$ | (j) $B \times A$                     |

2. List all the relationships between the sets in Part (1) that you observe.

One purpose of the work in Progress Checks 5.23 and 5.24 was to indicate the plausibility of many of the results contained in the next theorem.



**Theorem 5.25.** *Let  $A$ ,  $B$ , and  $C$  be sets. Then*

1.  $A \times (B \cap C) = (A \times B) \cap (A \times C)$
2.  $A \times (B \cup C) = (A \times B) \cup (A \times C)$
3.  $(A \cap B) \times C = (A \times C) \cap (B \times C)$
4.  $(A \cup B) \times C = (A \times C) \cup (B \times C)$
5.  $A \times (B - C) = (A \times B) - (A \times C)$
6.  $(A - B) \times C = (A \times C) - (B \times C)$
7. If  $T \subseteq A$ , then  $T \times B \subseteq A \times B$ .
8. If  $Y \subseteq B$ , then  $A \times Y \subseteq A \times B$ .

We will not prove all these results; rather, we will prove Part (2) of Theorem 5.25 and leave some of the rest to the exercises. In constructing these proofs, we need to keep in mind that Cartesian products are sets, and so we follow many of the same principles to prove set relationships that were introduced in Sections 5.2 and 5.3.

The other thing to remember is that the elements of a Cartesian product are ordered pairs. So when we start a proof of a result such as Part (2) of Theorem 5.25, the primary goal is to prove that the two sets are equal. We will do this by proving that each one is a subset of the other one. So if we want to prove that  $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ , we can start by choosing an arbitrary element of  $A \times (B \cup C)$ . The goal is then to show that this element must be in  $(A \times B) \cup (A \times C)$ . When we start by choosing an arbitrary element of  $A \times (B \cup C)$ , we could give that element a name. For example, we could start by letting

$$u \text{ be an element of } A \times (B \cup C). \quad (1)$$

We can then use the definition of “ordered pair” to conclude that

$$\text{there exists } x \in A \text{ and there exists } y \in B \cup C \text{ such that } u = (x, y). \quad (2)$$

In order to prove that  $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ , we must now show that the ordered pair  $u$  from (1) is in  $(A \times B) \cup (A \times C)$ . In order to do this, we can use the definition of set union and prove that

$$u \in (A \times B) \text{ or } u \in (A \times C). \quad (3)$$



Since  $u = (x, y)$ , we can prove (3) by proving that

$$(x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C). \quad (4)$$

If we look at the sentences in (2) and (4), it would seem that we are very close to proving that  $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ . Following is a proof of Part (2) of Theorem 5.25.

**Theorem 5.25** (Part (2)). *Let  $A$ ,  $B$ , and  $C$  be sets. Then*

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

**Proof.** Let  $A$ ,  $B$ , and  $C$  be sets. We will prove that  $A \times (B \cup C)$  is equal to  $(A \times B) \cup (A \times C)$  by proving that each set is a subset of the other set.

To prove that  $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ , we let  $u \in A \times (B \cup C)$ . Then there exists  $x \in A$  and there exists  $y \in B \cup C$  such that  $u = (x, y)$ . Since  $y \in B \cup C$ , we know that  $y \in B$  or  $y \in C$ .

In the case where  $y \in B$ , we have  $u = (x, y)$ , where  $x \in A$  and  $y \in B$ . So in this case,  $u \in A \times B$ , and hence  $u \in (A \times B) \cup (A \times C)$ . Similarly, in the case where  $y \in C$ , we have  $u = (x, y)$ , where  $x \in A$  and  $y \in C$ . So in this case,  $u \in A \times C$  and, hence,  $u \in (A \times B) \cup (A \times C)$ .

In both cases,  $u \in (A \times B) \cup (A \times C)$ . Hence, we may conclude that if  $u$  is an element of  $A \times (B \cup C)$ , then  $u \in (A \times B) \cup (A \times C)$ , and this proves that

$$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C). \quad (1)$$

We must now prove that  $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ . So we let  $v \in (A \times B) \cup (A \times C)$ . Then  $v \in (A \times B)$  or  $v \in (A \times C)$ .

In the case where  $v \in (A \times B)$ , we know that there exists  $s \in A$  and there exists  $t \in B$  such that  $v = (s, t)$ . But since  $t \in B$ , we know that  $t \in B \cup C$ , and hence  $v \in A \times (B \cup C)$ . Similarly, in the case where  $v \in (A \times C)$ , we know that there exists  $s \in A$  and there exists  $t \in C$  such that  $v = (s, t)$ . But because  $t \in C$ , we can conclude that  $t \in B \cup C$  and, hence,  $v \in A \times (B \cup C)$ .

In both cases,  $v \in A \times (B \cup C)$ . Hence, we may conclude that if  $v \in (A \times B) \cup (A \times C)$ , then  $v \in A \times (B \cup C)$ , and this proves that

$$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C). \quad (2)$$

The relationships in (1) and (2) prove that  $A \times (B \cup C) = (A \times B) \cup (A \times C)$ . ■



**Final Note.** The definition of an ordered pair in Preview Activity 2 may have seemed like a lengthy definition, but in some areas of mathematics, an even more formal and precise definition of “ordered pair” is needed. This definition is explored in Exercise (10).

## Exercises for Section 5.4

- \* 1. Let  $A = \{1, 2\}$ ,  $B = \{a, b, c, d\}$ , and  $C = \{1, a, b\}$ . Use the roster method to list all of the elements of each of the following sets:

- |                  |                                      |
|------------------|--------------------------------------|
| (a) $A \times B$ | (e) $A \times (B \cap C)$            |
| (b) $B \times A$ | (f) $(A \times B) \cap (A \times C)$ |
| (c) $A \times C$ | (g) $A \times \emptyset$             |
| (d) $A^2$        | (h) $B \times \{2\}$                 |

2. Sketch a graph of each of the following Cartesian products in the Cartesian plane.

- |                            |                                  |
|----------------------------|----------------------------------|
| (a) $[0, 2] \times [1, 3]$ | (e) $\mathbb{R} \times (2, 4)$   |
| (b) $(0, 2) \times (1, 3]$ | (f) $(2, 4) \times \mathbb{R}$   |
| (c) $[2, 3] \times \{1\}$  | (g) $\mathbb{R} \times \{-1\}$   |
| (d) $\{1\} \times [2, 3]$  | (h) $\{-1\} \times [1, +\infty)$ |

- \* 3. Prove Theorem 5.25, Part (1):  $A \times (B \cap C) = (A \times B) \cap (A \times C)$ .

- \* 4. Prove Theorem 5.25, Part (4):  $(A \cup B) \times C = (A \times C) \cup (B \times C)$ .

5. Prove Theorem 5.25, Part (5):  $A \times (B - C) = (A \times B) - (A \times C)$ .

6. Prove Theorem 5.25, Part (7): If  $T \subseteq A$ , then  $T \times B \subseteq A \times B$ .

7. Let  $A = \{1\}$ ,  $B = \{2\}$ , and  $C = \{3\}$ .

- Explain why  $A \times B \neq B \times A$ .
- Explain why  $(A \times B) \times C \neq A \times (B \times C)$ .

8. Let  $A$  and  $B$  be nonempty sets. Prove that  $A \times B = B \times A$  if and only if  $A = B$ .





9. Is the following proposition true or false? Justify your conclusion.

Let  $A$ ,  $B$ , and  $C$  be sets with  $A \neq \emptyset$ . If  $A \times B = A \times C$ , then  $B = C$ .

Explain where the assumption that  $A \neq \emptyset$  is needed.

### Explorations and Activities

10. **(A Set Theoretic Definition of an Ordered Pair)** In elementary mathematics, the notion of an ordered pair introduced at the beginning of this section will suffice. However, if we are interested in a formal development of the Cartesian product of two sets, we need a more precise definition of ordered pair. Following is one way to do this in terms of sets. This definition is credited to Kazimierz Kuratowski (1896 – 1980). Kuratowski was a famous Polish mathematician whose main work was in the areas of topology and set theory. He was appointed the Director of the Polish Academy of Sciences and served in that position for 19 years.

Let  $x$  be an element of the set  $A$ , and let  $y$  be an element of the set  $B$ . The **ordered pair**  $(x, y)$  is defined to be the set  $\{\{x\}, \{x, y\}\}$ . That is,

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

- (a) Explain how this definition allows us to distinguish between the ordered pairs  $(3, 5)$  and  $(5, 3)$ .
- (b) Let  $A$  and  $B$  be sets and let  $a, c \in A$  and  $b, d \in B$ . Use this definition of an ordered pair and the concept of set equality to prove that  $(a, b) = (c, d)$  if and only if  $a = c$  and  $b = d$ .

An **ordered triple** can be thought of as a single triple of objects, denoted by  $(a, b, c)$ , with an implied order. This means that in order for two ordered triples to be equal, they must contain exactly the same objects in the same order. That is,  $(a, b, c) = (p, q, r)$  if and only if  $a = p$ ,  $b = q$  and  $c = r$ .

- (c) Let  $A$ ,  $B$ , and  $C$  be sets, and let  $x \in A$ ,  $y \in B$ , and  $z \in C$ . Write a set theoretic definition of the ordered triple  $(x, y, z)$  similar to the set theoretic definition of “ordered pair.”

## 5.5 Indexed Families of Sets

### Preview Activity 1 (The Union and Intersection of a Family of Sets)

In Section 5.3, we discussed various properties of set operations. We will now focus on the associative properties for set union and set intersection. Notice that the definition of “set union” tells us how to form the union of two sets. It is the associative law that allows us to discuss the union of three sets. Using the associate law, if  $A$ ,  $B$ , and  $C$  are subsets of some universal set, then we can define  $A \cup B \cup C$  to be  $(A \cup B) \cup C$  or  $A \cup (B \cup C)$ . That is,

$$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C).$$

For this activity, the universal set is  $\mathbb{N}$  and we will use the following four sets:

$$A = \{1, 2, 3, 4, 5\}$$

$$C = \{3, 4, 5, 6, 7\}$$

$$B = \{2, 3, 4, 5, 6\}$$

$$D = \{4, 5, 6, 7, 8\}.$$

1. Use the roster method to specify the sets  $A \cup B \cup C$ ,  $B \cup C \cup D$ ,  $A \cap B \cap C$ , and  $B \cap C \cap D$ .
2. Use the roster method to specify each of the following sets. In each case, be sure to follow the order specified by the parentheses.

$$(a) (A \cup B \cup C) \cup D$$

$$(e) (A \cap B \cap C) \cap D$$

$$(b) A \cup (B \cup C \cup D)$$

$$(f) A \cap (B \cap C \cap D)$$

$$(c) A \cup (B \cup C) \cup D$$

$$(g) A \cap (B \cap C) \cap D$$

$$(d) (A \cup B) \cup (C \cup D)$$

$$(h) (A \cap B) \cap (C \cap D)$$

3. Based on the work in Part (2), does the placement of the parentheses matter when determining the union (or intersection) of these four sets? Does this make it possible to define  $A \cup B \cup C \cup D$  and  $A \cap B \cap C \cap D$ ?

We have already seen that the elements of a set may themselves be sets. For example, the power set of a set  $T$ ,  $\mathcal{P}(T)$ , is the set of all subsets of  $T$ . The phrase, “a set of sets” sounds confusing, and so we often use the terms **collection** and **family** when we wish to emphasize that the elements of a given set are themselves sets. We would then say that the power set of  $T$  is the family (or collection) of sets that are subsets of  $T$ .

One of the purposes of the work we have done so far in this activity was to show that it is possible to define the union and intersection of a family of sets.



**Definition.** Let  $\mathcal{C}$  be a family of sets. The **union of  $\mathcal{C}$**  is defined as the set of all elements that are in at least one of the sets in  $\mathcal{C}$ . We write

$$\bigcup_{X \in \mathcal{C}} X = \{x \in U \mid x \in X \text{ for some } X \in \mathcal{C}\}$$

The **intersection of  $\mathcal{C}$**  is defined as the set of all elements that are in all of the sets in  $\mathcal{C}$ . That is,

$$\bigcap_{X \in \mathcal{C}} X = \{x \in U \mid x \in X \text{ for all } X \in \mathcal{C}\}$$

For example, consider the four sets  $A$ ,  $B$ ,  $C$ , and  $D$  used earlier in this activity and the sets

$$S = \{5, 6, 7, 8, 9\} \quad \text{and} \quad T = \{6, 7, 8, 9, 10\}.$$

We can then consider the following families of sets:  $\mathcal{A} = \{A, B, C, D\}$  and  $\mathcal{B} = \{A, B, C, D, S, T\}$ .

4. Explain why

$$\bigcup_{X \in \mathcal{A}} X = A \cup B \cup C \cup D \quad \text{and} \quad \bigcap_{X \in \mathcal{A}} X = A \cap B \cap C \cap D,$$

and use your work in (1), (2), and (3) to determine  $\bigcup_{X \in \mathcal{A}} X$  and  $\bigcap_{X \in \mathcal{A}} X$ .

5. Use the roster method to specify  $\bigcup_{X \in \mathcal{B}} X$  and  $\bigcap_{X \in \mathcal{B}} X$ .

6. Use the roster method to specify the sets  $\left(\bigcup_{X \in \mathcal{A}} X\right)^c$  and  $\bigcap_{X \in \mathcal{A}} X^c$ . Remember that the universal set is  $\mathbb{N}$ .

### Preview Activity 2 (An Indexed Family of Sets)

We often use subscripts to identify sets. For example, in Preview Activity 1, instead of using  $A$ ,  $B$ ,  $C$ , and  $D$  as the names of the sets, we could have used  $A_1$ ,  $A_2$ ,  $A_3$ , and  $A_4$ . When we do this, we are using the subscript as an identifying tag, or index, for each set. We can also use this idea to specify an infinite family of sets. For example, for each natural number  $n$ , we define

$$C_n = \{n, n + 1, n + 2, n + 3, n + 4\}.$$



So if we have a family of sets  $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ , we use the notation  $\bigcup_{j=1}^4 C_j$  to mean the same thing as  $\bigcup_{X \in \mathcal{C}} X$ .

1. Determine  $\bigcup_{j=1}^4 C_j$  and  $\bigcap_{j=1}^4 C_j$ .

We can see that with the use of subscripts, we do not even have to define the family of sets  $\mathcal{A}$ . We can work with the infinite family of sets

$$\mathcal{C}^* = \{A_n \mid n \in \mathbb{N}\}$$

and use the subscripts to indicate which sets to use in a union or an intersection.

2. Use the roster method to specify each of the following pairs of sets. The universal set is  $\mathbb{N}$ .

(a)  $\bigcup_{j=1}^6 C_j$  and  $\bigcap_{j=1}^6 C_j$

(c)  $\bigcup_{j=4}^8 C_j$  and  $\bigcap_{j=4}^8 C_j$

(b)  $\bigcup_{j=1}^8 C_j$  and  $\bigcap_{j=1}^8 C_j$

(d)  $\left(\bigcap_{j=1}^4 C_j\right)^c$  and  $\bigcup_{j=1}^4 C_j^c$

### The Union and Intersection of an Indexed Family of Sets

One of the purposes of the preview activities was to show that we often encounter situations in which more than two sets are involved, and it is possible to define the union and intersection of more than two sets. In Preview Activity 2, we also saw that it is often convenient to “index” the sets in a family of sets. In particular, if  $n$  is a natural number and  $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$  is a family of  $n$  sets, then the union of these  $n$  sets, denoted by  $A_1 \cup A_2 \cup \dots \cup A_n$  or  $\bigcup_{j=1}^n A_j$ , is defined as

$$\bigcup_{j=1}^n A_j = \{x \in U \mid x \in A_j, \text{ for some } j \text{ with } 1 \leq j \leq n\}.$$



We can also define the intersection of these  $n$  sets, denoted by  $A_1 \cap A_2 \cap \cdots \cap A_n$  or  $\bigcap_{j=1}^n A_j$ , as

$$\bigcap_{j=1}^n A_j = \{x \in U \mid x \in A_j, \text{ for all } j \text{ with } 1 \leq j \leq n\}.$$

We can also extend this idea to define the union and intersection of a family that consists of infinitely many sets. So if  $\mathcal{B} = \{B_1, B_2, \dots, B_n, \dots\}$ , then

$$\bigcup_{j=1}^{\infty} B_j = \{x \in U \mid x \in B_j, \text{ for some } j \text{ with } j \geq 1\}, \text{ and}$$

$$\bigcap_{j=1}^{\infty} B_j = \{x \in U \mid x \in B_j, \text{ for all } j \text{ with } j \geq 1\}.$$

---

**Progress Check 5.26 (An Infinite Family of Sets)**

For each natural number  $n$ , let  $A_n = \{1, n, n^2\}$ . For example,

$$A_1 = \{1\}$$

$$A_2 = \{1, 2, 4\},$$

$$A_3 = \{1, 3, 9\},$$

and

$$\bigcup_{j=1}^3 A_j = \{1, 2, 3, 4, 9\},$$

$$\bigcap_{j=1}^3 A_j = \{1\}.$$

Determine each of the following sets:

1.  $\bigcup_{j=1}^6 A_j$

3.  $\bigcup_{j=3}^6 A_j$

5.  $\bigcup_{j=1}^{\infty} A_j$

2.  $\bigcap_{j=1}^6 A_j$

4.  $\bigcap_{j=3}^6 A_j$

6.  $\bigcap_{j=1}^{\infty} A_j$

---

In all of the examples we have studied so far, we have used  $\mathbb{N}$  or a subset of  $\mathbb{N}$  to index or label the sets in a family of sets. We can use other sets to index or label sets in a family of sets. For example, for each real number  $x$ , we can define  $B_x$  to be the closed interval  $[x, x + 2]$ . That is,

$$B_x = \{y \in \mathbb{R} \mid x \leq y \leq x + 2\}.$$



So we make the following definition. In this definition,  $\Lambda$  is the uppercase Greek letter lambda and  $\alpha$  is the lowercase Greek letter alpha.

**Definition.** Let  $\Lambda$  be a nonempty set and suppose that for each  $\alpha \in \Lambda$ , there is a corresponding set  $A_\alpha$ . The family of sets  $\{A_\alpha \mid \alpha \in \Lambda\}$  is called an **indexed family of sets** indexed by  $\Lambda$ . Each  $\alpha \in \Lambda$  is called an **index** and  $\Lambda$  is called an **indexing set**.

### Progress Check 5.27 (Indexed Families of Sets)

In each of the indexed families of sets that we seen so far, if the indices were different, then the sets were different. That is, if  $\Lambda$  is an indexing set for the family of sets  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$ , then if  $\alpha, \beta \in \Lambda$  and  $\alpha \neq \beta$ , then  $A_\alpha \neq A_\beta$ . (**Note:** The letter  $\beta$  is the Greek lowercase beta.)

1. Let  $\Lambda = \{1, 2, 3, 4\}$ , and for each  $n \in \Lambda$ , let  $A_n = \{2n + 6, 16 - 2n\}$ , and let  $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ . Determine  $A_1, A_2, A_3$ , and  $A_4$ .

2. Is the following statement true or false for the indexed family  $\mathcal{A}$  in (1)?

For all  $m, n \in \Lambda$ , if  $m \neq n$ , then  $A_m \neq A_n$ .

3. Now let  $\Lambda = \mathbb{R}$ . For each  $x \in \mathbb{R}$ , define  $B_x = \{0, x^2, x^4\}$ . Is the following statement true for the indexed family of sets  $\mathcal{B} = \{B_x \mid x \in \mathbb{R}\}$ ?

For all  $x, y \in \mathbb{R}$ , if  $x \neq y$ , then  $B_x \neq B_y$ .

We now restate the definitions of the union and intersection of a family of sets for an indexed family of sets.

**Definition.** Let  $\Lambda$  be a nonempty indexing set and let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets. The **union over**  $\mathcal{A}$  is defined as the set of all elements that are in at least one of sets  $A_\alpha$ , where  $\alpha \in \Lambda$ . We write

$$\bigcup_{\alpha \in \Lambda} A_\alpha = \{x \in U \mid \text{there exists an } \alpha \in \Lambda \text{ with } x \in A_\alpha\}.$$

The **intersection over**  $\mathcal{A}$  is the set of all elements that are in all of the sets  $A_\alpha$  for each  $\alpha \in \Lambda$ . That is,

$$\bigcap_{\alpha \in \Lambda} A_\alpha = \{x \in U \mid \text{for all } \alpha \in \Lambda, x \in A_\alpha\}.$$

**Example 5.28 (A Family of Sets Indexed by the Positive Real Numbers)**

For each positive real number  $\alpha$ , let  $A_\alpha$  be the interval  $(-1, \alpha]$ . That is,

$$A_\alpha = \{x \in \mathbb{R} \mid -1 < x \leq \alpha\}.$$

If we let  $\mathbb{R}^+$  be the set of positive real numbers, then we have a family of sets indexed by  $\mathbb{R}^+$ . We will first determine the union of this family of sets. Notice that for each  $\alpha \in \mathbb{R}^+$ ,  $\alpha \in A_\alpha$ , and if  $y$  is a real number with  $-1 < y \leq 0$ , then  $y \in A_\alpha$ . Also notice that if  $y \in \mathbb{R}$  and  $y < -1$ , then for each  $\alpha \in \mathbb{R}^+$ ,  $y \notin A_\alpha$ . With these observations, we conclude that

$$\bigcup_{\alpha \in \mathbb{R}^+} A_\alpha = (-1, \infty) = \{x \in \mathbb{R} \mid -1 < x\}.$$

To determine the intersection of this family, notice that

- if  $y \in \mathbb{R}$  and  $y < -1$ , then for each  $\alpha \in \mathbb{R}^+$ ,  $y \notin A_\alpha$ ;
- if  $y \in \mathbb{R}$  and  $-1 < y \leq 0$ , then  $y \in A_\alpha$  for each  $\alpha \in \mathbb{R}^+$ ; and
- if  $y \in \mathbb{R}$  and  $y > 0$ , then if we let  $\beta = \frac{y}{2}$ ,  $y > \beta$  and  $y \notin A_\beta$ .

From these observations, we conclude that

$$\bigcap_{\alpha \in \mathbb{R}^+} A_\alpha = (-1, 0] = \{x \in \mathbb{R} \mid -1 < x \leq 0\}.$$

**Progress Check 5.29 (A Continuation of Example 5.28)**

Using the family of sets from Example 5.28, for each  $\alpha \in \mathbb{R}^+$ , we see that

$$A_\alpha^c = (-\infty, -1] \cup (\alpha, \infty).$$

Use the results from Example 5.28 to help determine each of the following sets. For each set, use either interval notation or set builder notation.

1.  $\left( \bigcup_{\alpha \in \mathbb{R}^+} A_\alpha \right)^c$

3.  $\left( \bigcap_{\alpha \in \mathbb{R}^+} A_\alpha \right)^c$

2.  $\bigcap_{\alpha \in \mathbb{R}^+} A_\alpha^c$

4.  $\bigcup_{\alpha \in \mathbb{R}^+} A_\alpha^c$

## Properties of Union and Intersection

In Theorem 5.30, we will prove some properties of set operations for indexed families of sets. Some of these properties are direct extensions of corresponding properties for two sets. For example, we have already proved De Morgan's Laws for two sets in Theorem 5.20 on page 248. The work in the preview activities and Progress Check 5.29 suggests that we should get similar results using set operations with an indexed family of sets. For example, in Preview Activity 2, we saw that

$$\left( \bigcap_{j=1}^4 A_j \right)^c = \bigcup_{j=1}^4 A_j^c.$$

**Theorem 5.30.** *Let  $\Lambda$  be a nonempty indexing set and let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets, each of which is a subset of some universal set  $U$ . Then*

$$1. \text{ For each } \beta \in \Lambda, \bigcap_{\alpha \in \Lambda} A_\alpha \subseteq A_\beta.$$

$$2. \text{ For each } \beta \in \Lambda, A_\beta \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha.$$

$$3. \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right)^c = \bigcup_{\alpha \in \Lambda} A_\alpha^c$$

$$4. \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right)^c = \bigcap_{\alpha \in \Lambda} A_\alpha^c$$

Parts (3) and (4) are known as **De Morgan's Laws**.

**Proof.** We will prove Parts (1) and (3). The proofs of Parts (2) and (4) are included in Exercise (4). So we let  $\Lambda$  be a nonempty indexing set and let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets. To prove Part (1), we let  $\beta \in \Lambda$  and note that if  $x \in \bigcap_{\alpha \in \Lambda} A_\alpha$ , then  $x \in A_\alpha$ , for all  $\alpha \in \Lambda$ . Since  $\beta$  is one element in  $\Lambda$ , we may conclude that  $x \in A_\beta$ . This proves that  $\bigcap_{\alpha \in \Lambda} A_\alpha \subseteq A_\beta$ .

To prove Part (3), we will prove that each set is a subset of the other set. We first let  $x \in \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right)^c$ . This means that  $x \notin \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right)$ , and this means that





there exists a  $\beta \in \Lambda$  such that  $x \notin A_\beta$ .

Hence,  $x \in A_\beta^c$ , which implies that  $x \in \bigcup_{\alpha \in \Lambda} A_\alpha^c$ . Therefore, we have proved that

$$\left( \bigcap_{\alpha \in \Lambda} A_\alpha \right)^c \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha^c. \quad (1)$$

We now let  $y \in \bigcup_{\alpha \in \Lambda} A_\alpha^c$ . This means that there exists a  $\beta \in \Lambda$  such that  $y \in A_\beta^c$  or  $y \notin A_\beta$ . However, since  $y \notin A_\beta$ , we may conclude that  $y \notin \bigcap_{\alpha \in \Lambda} A_\alpha$  and, hence,  $y \in \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right)^c$ . This proves that

$$\bigcup_{\alpha \in \Lambda} A_\alpha^c \subseteq \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right)^c. \quad (2)$$

Using the results in (1) and (2), we have proved that  $\left( \bigcap_{\alpha \in \Lambda} A_\alpha \right)^c = \bigcup_{\alpha \in \Lambda} A_\alpha^c$ . ■

Many of the other properties of set operations are also true for indexed families of sets. Theorem 5.31 states the **distributive laws** for set operations.

**Theorem 5.31.** *Let  $\Lambda$  be a nonempty indexing set, let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets, and let  $B$  be a set. Then*

1.  $B \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha)$ , and
2.  $B \cup \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right) = \bigcap_{\alpha \in \Lambda} (B \cup A_\alpha)$ .

The proof of Theorem 5.31 is Exercise (5).

### Pairwise Disjoint Families of Sets

In Section 5.2, we defined two sets  $A$  and  $B$  to be disjoint provided that  $A \cap B = \emptyset$ . In a similar manner, if  $\Lambda$  is a nonempty indexing set and  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  is an indexed family of sets, we can say that this indexed family of sets is **disjoint**



provided that  $\bigcap_{\alpha \in \Lambda} A_\alpha = \emptyset$ . However, we can use the concept of two disjoint sets to define a somewhat more interesting type of “disjointness” for an indexed family of sets.

**Definition.** Let  $\Lambda$  be a nonempty indexing set, and let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets. We say that  $\mathcal{A}$  is **pairwise disjoint** provided that for all  $\alpha$  and  $\beta$  in  $\Lambda$ , if  $A_\alpha \neq A_\beta$ , then  $A_\alpha \cap A_\beta = \emptyset$ .

**Progress Check 5.32 (Disjoint Families of Sets)**

Figure 5.7 shows two families of sets,

$$\mathcal{A} = \{A_1, A_2, A_3, A_4\} \text{ and } \mathcal{B} = \{B_1, B_2, B_3, B_4\}.$$

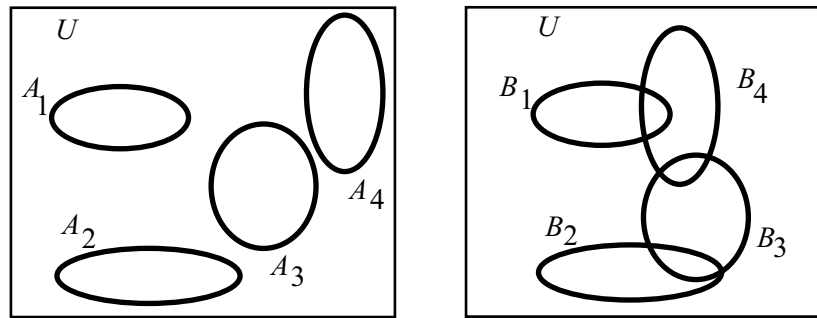


Figure 5.7: Two Families of Indexed Sets

1. Is the family of sets  $\mathcal{A}$  a disjoint family of sets? A pairwise disjoint family of sets?
2. Is the family of sets  $\mathcal{B}$  a disjoint family of sets? A pairwise disjoint family of sets?

Now let the universal set be  $\mathbb{R}$ . For each  $n \in \mathbb{N}$ , let  $C_n = (n, \infty)$ , and let  $\mathcal{C} = \{C_n \mid n \in \mathbb{N}\}$ .

3. Is the family of sets  $\mathcal{C}$  a disjoint family of sets? A pairwise disjoint family of sets?

## Exercises for Section 5.5

1. For each natural number  $n$ , let  $A_n = \{n, n+1, n+2, n+3\}$ . Use the roster method to specify each of the following sets:

$$\star \text{ (a) } \bigcap_{j=1}^3 A_j$$

$$\star \text{ (d) } \bigcup_{j=3}^7 A_j$$

$$\text{ (b) } \bigcup_{j=1}^3 A_j$$

$$\text{ (e) } A_9 \cap \left( \bigcup_{j=3}^7 A_j \right)$$

$$\text{ (c) } \bigcap_{j=3}^7 A_j$$

$$\text{ (f) } \bigcup_{j=3}^7 (A_9 \cap A_j)$$

2. For each natural number  $n$ , let  $A_n = \{k \in \mathbb{N} \mid k \geq n\}$ . Assuming the universal set is  $\mathbb{N}$ , use the roster method or set builder notation to specify each of the following sets:

$$\star \text{ (a) } \bigcap_{j=1}^5 A_j$$

$$\text{ (e) } \bigcup_{j=1}^5 A_j$$

$$\text{ (b) } \left( \bigcap_{j=1}^5 A_j \right)^c$$

$$\star \text{ (f) } \left( \bigcup_{j=1}^5 A_j \right)^c$$

$$\star \text{ (c) } \bigcap_{j=1}^5 A_j^c$$

$$\text{ (g) } \bigcap_{j \in \mathbb{N}} A_j$$

$$\star \text{ (d) } \bigcup_{j=1}^5 A_j^c$$

$$\text{ (h) } \bigcup_{j \in \mathbb{N}} A_j$$

3. For each positive real number  $r$ , define  $T_r$  to be the closed interval  $[-r^2, r^2]$ . That is,

$$T_r = \{x \in \mathbb{R} \mid -r^2 \leq x \leq r^2\}.$$

Let  $\Lambda = \{m \in \mathbb{N} \mid 1 \leq m \leq 10\}$ . Use either interval notation or set builder notation to specify each of the following sets:

$$\star \text{ (a) } \bigcup_{k \in \Lambda} T_k$$

$$\text{ (c) } \bigcup_{r \in \mathbb{R}^+} T_r$$

$$\text{ (e) } \bigcup_{k \in \mathbb{N}} T_k$$

$$\star \text{ (b) } \bigcap_{k \in \Lambda} T_k$$

$$\text{ (d) } \bigcap_{r \in \mathbb{R}^+} T_r$$

$$\text{ (f) } \bigcap_{k \in \mathbb{N}} T_k$$



4. Prove Parts (2) and (4) of Theorem 5.30. Let  $\Lambda$  be a nonempty indexing set and let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets.

\* (a) For each  $\beta \in \Lambda$ ,  $A_\beta \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha$ .

(b)  $\left( \bigcup_{\alpha \in \Lambda} A_\alpha \right)^c = \bigcap_{\alpha \in \Lambda} A_\alpha^c$

5. Prove Theorem 5.31. Let  $\Lambda$  be a nonempty indexing set, let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets, and let  $B$  be a set. Then

\* (a)  $B \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha)$ , and

(b)  $B \cup \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right) = \bigcap_{\alpha \in \Lambda} (B \cup A_\alpha)$ .

6. Let  $\Lambda$  and  $\Gamma$  be nonempty indexing sets. (**Note:** The letter  $\Gamma$  is the uppercase Greek letter gamma.) Also, let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  and  $\mathcal{B} = \{B_\beta \mid \beta \in \Gamma\}$  be indexed families of sets. Use the distributive laws in Exercise (5) to:

(a) Write  $\left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) \cap \left( \bigcup_{\beta \in \Gamma} B_\beta \right)$  as a union of intersections of two sets.

(b) Write  $\left( \bigcap_{\alpha \in \Lambda} A_\alpha \right) \cup \left( \bigcap_{\beta \in \Gamma} B_\beta \right)$  as an intersection of unions of two sets.

7. Let  $\Lambda$  be a nonempty indexing set and let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets. Also, assume that  $\Gamma \subseteq \Lambda$  and  $\Gamma \neq \emptyset$ . Prove that

(a)  $\bigcup_{\alpha \in \Gamma} A_\alpha \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha$

(b)  $\bigcap_{\alpha \in \Lambda} A_\alpha \subseteq \bigcap_{\alpha \in \Gamma} A_\alpha$

8. Let  $\Lambda$  be a nonempty indexing set and let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets.

\* (a) Prove that if  $B$  is a set such that  $B \subseteq A_\alpha$  for every  $\alpha \in \Lambda$ , then  $B \subseteq \bigcap_{\alpha \in \Lambda} A_\alpha$ .

(b) Prove that if  $C$  is a set such that  $A_\alpha \subseteq C$  for every  $\alpha \in \Lambda$ , then  $\bigcup_{\alpha \in \Lambda} A_\alpha \subseteq C$ .

9. For each natural number  $n$ , let  $A_n = \{x \in \mathbb{R} \mid n - 1 < x < n\}$ . Prove that  $\{A_n \mid n \in \mathbb{N}\}$  is a pairwise disjoint family of sets and that  $\bigcup_{n \in \mathbb{N}} A_n = (\mathbb{R}^+ - \mathbb{N})$ .
10. For each natural number  $n$ , let  $A_n = \{k \in \mathbb{N} \mid k \geq n\}$ . Determine if the following statements are true or false. Justify each conclusion.
- (a) For all  $j, k \in \mathbb{N}$ , if  $j \neq k$ , then  $A_j \cap A_k \neq \emptyset$ .
- (b)  $\bigcap_{k \in \mathbb{N}} A_k = \emptyset$
11. Give an example of an indexed family of sets  $\{A_n \mid n \in \mathbb{N}\}$  such all three of the following conditions are true:
- (i) For each  $m \in \mathbb{N}$ ,  $A_m \subseteq (0, 1)$ ;
- (ii) For each  $j, k \in \mathbb{N}$ , if  $j \neq k$ , then  $A_j \cap A_k \neq \emptyset$ ; and
- (iii)  $\bigcap_{k \in \mathbb{N}} A_k = \emptyset$ .
12. Let  $\Lambda$  be a nonempty indexing set, let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets, and let  $B$  be a set. Use the results of Theorem 5.30 and Theorem 5.31 to prove each of the following:
- \* (a)  $\left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) - B = \bigcup_{\alpha \in \Lambda} (A_\alpha - B)$
- (b)  $\left( \bigcap_{\alpha \in \Lambda} A_\alpha \right) - B = \bigcap_{\alpha \in \Lambda} (A_\alpha - B)$
- (c)  $B - \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcap_{\alpha \in \Lambda} (B - A_\alpha)$
- (d)  $B - \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (B - A_\alpha)$

### Explorations and Activities

13. **An Indexed Family of Subsets of the Cartesian Plane.** Let  $\mathbb{R}^*$  be the set of nonnegative real numbers, and for each  $r \in \mathbb{R}^*$ , let

$$\begin{aligned} C_r &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = r^2\} \\ D_r &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 \leq r^2\} \\ T_r &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 > r^2\} = D_r^c. \end{aligned}$$



If  $r > 0$ , then the set  $C_r$  is the circle of radius  $r$  with center at the origin as shown in Figure 5.8, and the set  $D_r$  is the shaded disk (including the boundary) shown in Figure 5.8.

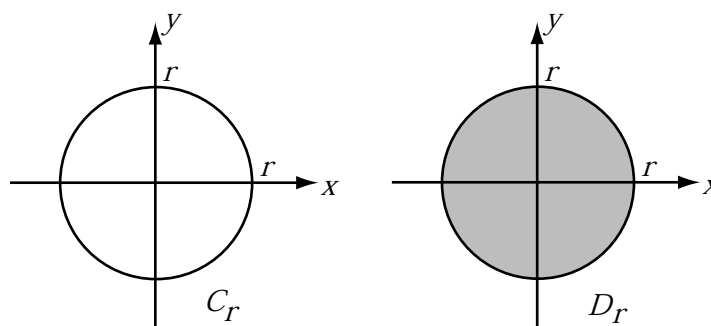


Figure 5.8: Two Sets for Activity 13

- (a) Determine  $\bigcup_{r \in \mathbb{R}^*} C_r$  and  $\bigcap_{r \in \mathbb{R}^*} C_r$ .
- (b) Determine  $\bigcup_{r \in \mathbb{R}^*} D_r$  and  $\bigcap_{r \in \mathbb{R}^*} D_r$ .
- (c) Determine  $\bigcup_{r \in \mathbb{R}^*} T_r$  and  $\bigcap_{r \in \mathbb{R}^*} T_r$ .
- (d) Let  $\mathcal{C} = \{C_r \mid r \in \mathbb{R}^*\}$ ,  $\mathcal{D} = \{D_r \mid r \in \mathbb{R}^*\}$ , and  $\mathcal{T} = \{T_r \mid r \in \mathbb{R}^*\}$ . Are any of these indexed families of sets pairwise disjoint? Explain.

Now let  $I$  be the closed interval  $[0, 2]$  and let  $J$  be the closed interval  $[1, 2]$ .

- (e) Determine  $\bigcup_{r \in I} C_r$ ,  $\bigcap_{r \in I} C_r$ ,  $\bigcup_{r \in J} C_r$ , and  $\bigcap_{r \in J} C_r$ .
- (f) Determine  $\bigcup_{r \in I} D_r$ ,  $\bigcap_{r \in I} D_r$ ,  $\bigcup_{r \in J} D_r$ , and  $\bigcap_{r \in J} D_r$ .
- (g) Determine  $\left(\bigcup_{r \in I} D_r\right)^c$ ,  $\left(\bigcap_{r \in I} D_r\right)^c$ ,  $\left(\bigcup_{r \in J} D_r\right)^c$ , and  $\left(\bigcap_{r \in J} D_r\right)^c$ .
- (h) Determine  $\bigcup_{r \in I} T_r$ ,  $\bigcap_{r \in I} T_r$ ,  $\bigcup_{r \in J} T_r$ , and  $\bigcap_{r \in J} T_r$ .
- (i) Use De Morgan's Laws to explain the relationship between your answers in Parts (13g) and (13h).

## 5.6 Chapter 5 Summary

### Important Definitions

- Equal sets, page 55
- Subset, page 55
- Proper subset, page 218
- Power set, page 222
- Cardinality of a finite set, page 223
- Intersection of two sets, page 216
- Union of two sets, page 216
- Set difference, page 216
- Complement of a set, page 216
- Disjoint sets, page 236
- Cartesian product of two sets, pages 256
- Ordered pair, page 256
- Union over a family of sets, page 265
- Intersection over a family of sets, page 265
- Indexing set, page 268
- Indexed family of sets, page 268
- Union over an indexed family of sets, page 268
- Intersection over an indexed family of sets, page 268
- Pairwise disjoint family of sets, page 272

### Important Theorems and Results about Sets

- **Theorem 5.5.** *Let  $n$  be a nonnegative integer and let  $A$  be a subset of some universal set. If  $A$  is a finite set with  $n$  elements, then  $A$  has  $2^n$  subsets. That is, if  $|A| = n$ , then  $|\mathcal{P}(A)| = 2^n$ .*
- **Theorem 5.18.** *Let  $A$ ,  $B$ , and  $C$  be subsets of some universal set  $U$ . Then all of the following equalities hold.*

<i>Properties of the Empty Set and the Universal Set</i>	$A \cap \emptyset = \emptyset$	$A \cap U = A$
	$A \cup \emptyset = A$	$A \cup U = U$
<i>Idempotent Laws</i>	$A \cap A = A$	$A \cup A = A$
<i>Commutative Laws</i>	$A \cap B = B \cap A$	$A \cup B = B \cup A$
<i>Associative Laws</i>	$(A \cap B) \cap C = A \cap (B \cap C)$	
	$(A \cup B) \cup C = A \cup (B \cup C)$	



*Distributive Laws*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- **Theorem 5.20.** Let  $A$  and  $B$  be subsets of some universal set  $U$ . Then the following are true:

*Basic Properties*

$$(A^c)^c = A$$

$$A - B = A \cap B^c$$

*Empty Set, Universal Set*

$$A - \emptyset = A \text{ and } A - U = \emptyset$$

$$\emptyset^c = U \text{ and } U^c = \emptyset$$

*De Morgan's Laws*

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$

*Subsets and Complements*

$$A \subseteq B \text{ if and only if } B^c \subseteq A^c.$$

- **Theorem 5.25.** Let  $A$ ,  $B$ , and  $C$  be sets. Then

$$1. A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$2. A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$3. (A \cap B) \times C = (A \times C) \cap (B \times C)$$

$$4. (A \cup B) \times C = (A \times C) \cup (B \times C)$$

$$5. A \times (B - C) = (A \times B) - (A \times C)$$

$$6. (A - B) \times C = (A \times C) - (B \times C)$$

$$7. \text{ If } T \subseteq A, \text{ then } T \times B \subseteq A \times B.$$

$$8. \text{ If } Y \subseteq B, \text{ then } A \times Y \subseteq A \times B.$$

- **Theorem 5.30.** Let  $\Lambda$  be a nonempty indexing set and let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets. Then

$$1. \text{ For each } \beta \in \Lambda, \bigcap_{\alpha \in \Lambda} A_\alpha \subseteq A_\beta.$$

$$2. \text{ For each } \beta \in \Lambda, A_\beta \subseteq \bigcap_{\alpha \in \Lambda} A_\alpha.$$

$$3. \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right)^c = \bigcup_{\alpha \in \Lambda} A_\alpha^c$$

$$4. \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right)^c = \bigcap_{\alpha \in \Lambda} A_\alpha^c$$

Parts (3) and (4) are known as **De Morgan's Laws**.





- **Theorem 5.31.** *Let  $\Lambda$  be a nonempty indexing set, let  $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$  be an indexed family of sets, and let  $B$  be a set. Then*

$$\begin{aligned} 1. \quad & B \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha), \text{ and} \\ 2. \quad & B \cup \left( \bigcap_{\alpha \in \Lambda} A_\alpha \right) = \bigcap_{\alpha \in \Lambda} (B \cup A_\alpha). \end{aligned}$$

### Important Proof Method

#### The Choose-an-Element Method

The choose-an-element method is frequently used when we encounter a universal quantifier in a statement in the backward process of a proof. This statement often has the form

For each element with a given property, something happens.

In the forward process of the proof, we then we choose an arbitrary element with the given property.

*Whenever we choose an arbitrary element with a given property, we are not selecting a specific element. Rather, the only thing we can assume about the element is the given property.*

For more information, see page [232](#).

