Sets, relations and graphs

Set theory is important, because it serves as a foundation for the rest of mathematics.

Sets are simply collections of objects. The objects belonging to a set are called **members** or **elements**. The elements in a set are *unordered*. Essential is only whether an object is in a set or not.

The easiest way to define a set is to enumerate its **elements** – inside curly brackets. For example $A = \{1, 2, 3\}$ is a set, and so is $B = \{\heartsuit, \spadesuit\}$. Here A and B are **names** of a set.

The members of sets must all be *distinct objects*. For example $A = \{a, b, a\}$ is not a set, because a is in A twice. $\{b, a\}$ is the same set as $\{a, b\}$.

Let us introduce the following notation used in set theory:

- $x \in A$ (the element x **belongs** to the set A)
- $x \notin A$ (x is **not** an element of A)
- $A \subseteq B$ (A is a **subset** of B: if $x \in A$, then $x \in B$)
- $A \nsubseteq B$ (A is **not** a subset of B)
- $A = B$ (A and B are the **same** set; $A \subseteq B$ and $B \subseteq A$)
- $A \neq B$ (A and B are **different** sets)
- $A \subset B$ (A is a **proper** subset of B; $A \subseteq B$, but $A \neq B$)
- $A \supseteq B$ ($B \subseteq A$)
- $A \supset B$ ($B \subset A$)
- $\emptyset$ (the **empty set**).

The empty set $\emptyset$ is a set with no elements. It can be also written $\{\}$.

Let us now illustrate these:

Elements of a set: If $A = \{a, b, d, h, i, j, l\}$, then $a \in A$, $c \notin A$, $h \in A$.

Subsets:

$$\{a, b\} \subseteq \{a, b, c\} \text{ and } \{a, b, c\} \supseteq \{a, b\}$$
$$\{a, b\} \subset \{a, b, c\} \text{ and } \{a, b, c\} \supset \{a, b\}$$
$$\{a, b\} \neq \{a, b, c\}$$
$$\{a, b, d\} \nsubseteq \{a, b, c\}$$

The empty set $\emptyset$ is a subset of every set. Note also $\emptyset \neq \{\emptyset\}$. Why?

Set builder: Another way to define sets is to give a condition defining the set in question:

$$\{x \mid C(x)\}$$

denotes the set of elements satisfying the condition C. For instance, let

$$S = \{x \mid x \text{ is an odd positive integer smaller than } 1000\}.$$

This means that the set S is $\{1, 3, 5, 7, \ldots, 999\}$

It is also normal to show what "type" T of elements a set consists of:

$$\{x \in T \mid C(x)\}.$$

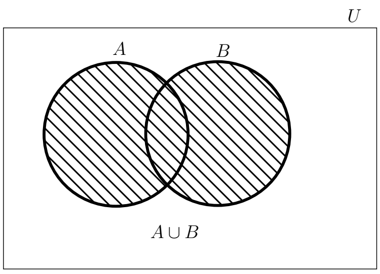
We will consider this in detail a little later.

A **Venn diagram** is an illustration that uses circles to show the relationships between sets. Often we consider the subsets of some **universe** U – consisting of the elements under consideration for a particular problem or situation. The universe U is represented by a rectangle and its subsets are denoted by circles.

One may form new sets from existing sets by using the set-theoretical operations:

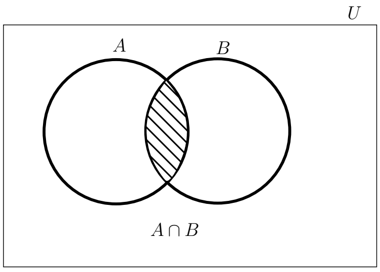
- **Union** of the sets A and B, denoted $A \cup B$, is the set of all objects that are a member of A or B, or both. For example, the union $\{1, 2, 3\} \cup \{2, 3, 4\}$ is the set $\{1, 2, 3, 4\}$. The union can be expressed as:

$$A \cup B = \{a \mid a \in A \text{ or } a \in B\}.$$



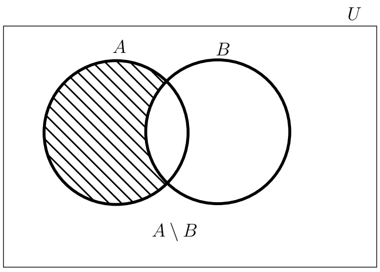
- **Intersection** of the sets A and B, denoted $A \cap B$, is the set of all objects that are members of both A and B. For example, the intersection of $\{1, 2, 3\} \cap \{2, 3, 4\}$ is the set $\{2, 3\}$. The intersection can be expressed as:

$$A \cap B = \{a \mid a \in A \text{ and } a \in B\}.$$

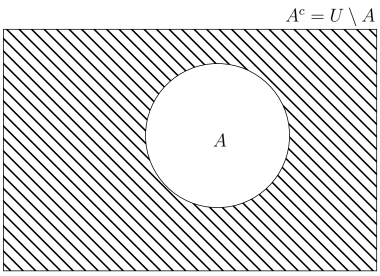


- **Set difference** of A and B, denoted $A \setminus B$ or $A - B$, is the set of all members of A that are not members of B. The set difference $\{1, 2, 3\} \setminus \{2, 3, 4\}$ is $\{1\}$, while the set difference $\{2, 3, 4\} \setminus \{1, 2, 3\}$ is $\{4\}$. The difference is expressed as:

$$A \setminus B = \{a \mid a \in A \text{ and } a \notin B\}.$$



- For $A \subseteq U$, the difference $U \setminus A$ is also called the **complement** of A in U, denoted by A^c.



- **Symmetric difference** of the sets A and B contains the elements that are in exactly one of A and B. The symmetric difference is denoted by This is denoted by $A \ominus B$ or $A \bigtriangleup B$. Note that $A \ominus B = (A \cup B) \setminus (A \cap B)$.

The **power set** of a set U, denoted $\wp(U)$ is the set whose members are all subsets of U. If $U = \{a, b, c\}$, then

$$\wp(U) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}.$$

Moreover, $\wp(\emptyset) = \{\emptyset\}$.

In mathematics, results and statements are often presented in the form of proposition and theorems. The word **Theorem** is usually reserved for most important results. **Proposition** is also an interesting result, but generally less important than a theorem. In addition, there are lemmas. A **Lemma** is a minor result whose sole purpose is to help in proving a theorem or a proposition.

A **formal proof** is required to show the correctness of a claim. We consider formal proofs in detail a little bit later in this course, but we consider proofs here briefly. A mathematical proof shows that the stated assumptions logically guarantee the conclusion. In our first proposition, we only assume that A, B and C are some subsets of the universe U. The proposition contains separate claims (a), (b), (c) and (d).

Proposition 1. *Let U be a set and $A, B, C \subseteq U$. The following equations are true.*

(1) $A \cap (A \cup B) = A$ *and* $A \cup (A \cap B) = A$.

(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(3) $A \cup A^c = U$ *and* $A \cap A^c = \emptyset$.

(4) $(A \cap B)^c = A^c \cup B^c$ *and* $(A \cup B)^c = A^c \cap B^c$.

The validity of the claims of the above proposition can be seen in Venn diagrams. But let us also provide a formal proof. All these statements say that some sets are the same (using the symbol "="). Showing that sets A and B are equal can be done by showing $A \subseteq B$ and $B \subseteq A$. Additionally, $A \subseteq B$ can be sometimes shown easier by showing that $B^c \subseteq A^c$.

Proof. (1) If $x \in A \cap (A \cup B)$, then $x \in A$ and $x \in (A \cup B)$ by the definition of intersection. In particular, $x \in A$. Thus, $A \cap (A \cup B) \subseteq A$. The other direction is similar: if $x \in A$, then $x \in A$ and $x \in A \subseteq (A \cup B)$, so $x \in A \cap (A \cup B)$. Hence, $A \subseteq A \cap (A \cup B)$.

The claim $A \cup (A \cap B) = A$ can be proved in a similar way.

(3) Because every element of U belongs to either A or its complement A^c, the claims follows immediately.

Cases (2) and (4) are left for exercises. $\square$

Proofs end often with an open box $\square$, which is there just for clarity.

A collection $\mathcal{F}$ of subsets of a given set U is called a **family of subsets** of U. For instance, the power set $\wp(U)$ is a family of sets.

Let $\mathcal{F}$ be a family of subsets of U. The union and intersection of $\mathcal{F}$ are defined by:

Union: $\bigcup \mathcal{F} = \{x \in U \mid \text{there is } A \in \mathcal{F} \text{ such that } x \in A\}$

The set of the elements that belong at least one set of $\mathcal{F}$.

Intersection: $\bigcap \mathcal{F} = \{x \in U \mid x \in A \text{ for all } A \in \mathcal{F}\ \}$

The set of the elements that belong to all sets of $\mathcal{F}$.

Example 2. Let $U = \{1, 2, 3, 4, 5\}$ and $\mathcal{F} = \{A_1, A_2, A_3\}$, where

$$A_1 = \{1, 2, 3\}, \qquad A_2 = \{2, 5\}, \qquad A_3 = \{2, 3, 5\}.$$

$$\bigcup \mathcal{F} = A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 5\} \quad \text{and} \quad \bigcap \mathcal{F} = A_1 \cap A_2 \cap A_3 = \{2\}.$$

Because also $\emptyset$ is a family of set, we should define $\bigcup \emptyset$ and $\bigcap \emptyset$. It is natural to state that

$$\bigcup \emptyset = \emptyset.$$

Indeed, if $\mathcal{F} = \emptyset$, then for any $x \in U$, $x \in \bigcup \mathcal{F}$ means that there is at least one $A \in \mathcal{F}$ such that $x \in A$. But because $\mathcal{F}$ is empty, there is no such set. Therefore, $\bigcup \emptyset = \emptyset$.

On the other hand, an element $x \in U$ is in $\bigcap \mathcal{F}$ if x belongs to *all* sets in $\mathcal{F}$. Because there is no sets in $\mathcal{F}$, the element x belongs to all of them (so-called *vacuously true*). Thus, $\bigcap \emptyset = U$. The other way to deal with this would be to consider the situation in which $\bigcap \emptyset \neq U$. This would mean that there is an element $x \in U$ such that $x \notin \bigcap \emptyset$. But this would require that there is a set A such that $x \notin A$ and $A \in \mathcal{F} = \emptyset$. But since the family $\mathcal{F}$ is empty, there is no such set A. Therefore, we can write

$$\bigcap \emptyset = U.$$

It may seem strange that the value of $\bigcap \emptyset$ depends on the universe U, because there is only one $\emptyset$. If you search internet, you may see that some people have the opinion that $\bigcap \emptyset$ should not be defined – similarly as division by zero is not defined.

Lemma 3. *Let U be a set and let $\mathcal{G}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ be families of subsets of U.*

(a) *If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $\bigcup \mathcal{F}_1 \subseteq \bigcup \mathcal{F}_2$.*

(b) *If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $\bigcap \mathcal{F}_2 \subseteq \bigcap \mathcal{F}_1$.*

(c) $(\bigcap \mathcal{G})^c = \bigcup \{A^c \mid A \in \mathcal{G}\}$.

(d) $(\bigcup \mathcal{G})^c = \bigcap \{A^c \mid A \in \mathcal{G}\}$.

Proof. The proof is left as exercises. $\square$

If $\mathcal{F}_1 = \emptyset$, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$ holds for all families $\mathcal{F}_2 \subseteq \wp(U)$. Now Lemma 3 gives that $\bigcup \emptyset = \emptyset \subseteq \bigcup \mathcal{F}_2$ and $\bigcap \mathcal{F}_2 \subseteq \bigcap \mathcal{F}_1 = U$.

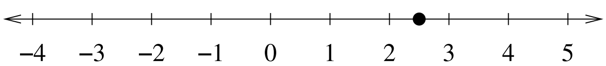
Some special sets have a fixed symbol to denote them:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$ (natural numbers)
- $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ (positive natural numbers)
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (integers)
- $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ (rational numbers)
- $\mathbb{R}$ (real numbers)
- $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ (complex numbers)

The following inclusions then hold:

$$\mathbb{N}_+ \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

The set $\mathbb{N}$ is equipped with ordinary addition and multiplication, for which $\mathbb{N}$ is **closed**: if $a, b \in \mathbb{N}$, then $a + b, ab \in \mathbb{N}$.
In order to perform subtractions, the set $\mathbb{N}$ must expands to $\mathbb{Z}$.
To enable a division, the set $\mathbb{Z}$ needs to be expanded to $\mathbb{Q}$.
We will show later that the real number $\sqrt{2}$ is not rational. A real number is any number that can be placed on a number line that extends to infinity in both the positive and negative directions. This number line is illustrated below with the number 2.5 marked with a closed dot as an example.



The above illustration, of course, only shows a portion of the number line (it would be impossible to show the whole thing), and only certain numbers are labeled $(-1, 0, 1, 2, \ldots)$. A real number can thus be 8, 4.357, $\frac{3}{7}$, π and any other such number. The particular representation, whether it be a fraction or a decimal does not matter. Perhaps the best way to describe a real number is to identify numbers that are not real numbers. Infinity (∞) is not a real number, but it is greater than any given real number. Infinity is not defined as a real number, because the set of real numbers should be closed under addition: if you add two real numbers together you will always get a real number. However, ∞ does not behave well with respect to addition.

Also the **imaginary unit** i is not a real number. The unit i is defined by its property $i^2 = -1$.

A **complex number** is a number that can be expressed in the form $a + bi$, where a and b are real numbers. Thus, $3i$, $2 + 5.4i$, and πi are all complex numbers. In fact, the real numbers can be seen as a subset of the complex numbers: any real number r can be written as $r + 0i$, which is a complex representation. Complex numbers allow solutions to all polynomial equations, even those that have no solutions in real numbers (this is not proved in this course). For example, the equation $(x + 1)^2 = -9$ has no solution in $\mathbb{R}$, since the square of a real number cannot be negative, but has the two non-real complex number solutions $-1 + 3i$ and $-1 - 3i$.

An **ordered pair** (a, b) is a pair of objects. Note that we use round brackets to denote ordered pairs. Curly brackets are used to denote sets. The type of brackets is important. The order in which the objects appear in the pair is significant: the ordered pair (a, b) is different from the ordered pair (b, a) – unless $a = b$. Recall that in sets, the order was not important: $\{a, b\} = \{b, a\}$.

The ordered pairs are equal if they have the same first component and the same second component, that is, $(a, b) = (c, d)$ whenever $a = c$ and $b = d$.

The set of all ordered pairs whose first entry is in some set A and whose second entry is in some set B is called the **Cartesian product** of A and B, and written $A \times B$, that is:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

The cartesian product $A \times A$ is often denoted by A^2.

Example 4. (a) $\{a, b\} \times \{a, c, d\} = \{(a, a), (a, c), (a, d), (b, a), (b, c), (b, d)\}$.

(b) Also a complex number $a + bi$ can be viewed as an ordered pair (a, b) and $\mathbb{C}$ can be identified with $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Example 5. An illustrative example is the standard 52-card deck:

- Ranks $= \{\mathrm{A}, \mathrm{K}, \mathrm{Q}, \mathrm{J}, 10, 9, 8, 7, 6, 5, 4, 3, 2\}$ form a 13-element set.
- Suits $= \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$ form a four-element set.

The Cartesian products Ranks $\times$ Suits and Suits $\times$ Ranks of these sets returns a 52-element set consisting of 52 ordered pairs, which correspond to all 52 possible playing cards.

- Ranks $\times$ Suits consists of pairs of the form:

$$\{(\mathrm{A}, \spadesuit), (\mathrm{A}, \clubsuit), (\mathrm{A}, \heartsuit), (\mathrm{A}, \diamondsuit), \ldots, (2, \spadesuit), (2, \clubsuit), (2, \heartsuit), (2, \diamondsuit)\}$$

- Suits $\times$ Ranks consists of pairs of the form:

$$\{(\heartsuit, \mathrm{A}), (\heartsuit, \mathrm{K}), \ldots, (\heartsuit, 2), (\diamondsuit, \mathrm{A}), (\diamondsuit, \mathrm{K}), \ldots, (\diamondsuit, 2),$$
$$(\clubsuit, \mathrm{A}), (\clubsuit, \mathrm{K}), \ldots, (\clubsuit, 2), (\spadesuit, \mathrm{A}), (\spadesuit, \mathrm{K}), \ldots, (\spadesuit, 2)\}$$

These two sets are *disjoint*, that is, they have no common element.

Note also that since Cartesian products are sets, the order of the pairs is not essential, but inside the pairs the order is important!

The Cartesian product can be generalized to the n**-ary Cartesian product** over n sets $X_1, \ldots, X_n$ as the set

$$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) \mid x_i \in A_i \text{ for every } i \in \{1, \ldots, n\}\}$$

The elements of the Cartesian product are called also n**-vectors**.

If $A_1 = \cdots = A_n = A$, then $A_1 \times \ldots \times A_n$ is n**-ary Cartesian power** of A, denoted by A^n. For instance, the coordinate systems $\mathbb{R}^2$ and $\mathbb{R}^3$ are examples of Cartesian powers.

A **binary relation** R over sets X and Y is a subset of the Cartesian product $X \times Y$, that is, $R \subseteq X \times Y$. This means that the relation R is a *set* consisting of ordered pairs (a, b) such that $a \in X$ and $b \in Y$. If R is a binary relation and $(a, b) \in R$, this is often denoted by $a\,R\,b$. For instance,

- $a = b$
- $3 > -1$
- $5 \leq 5$
- $A \subseteq B$

are such instances.

If x and y are *not* R-related, this is written $(x, y) \notin R$. In some cases we write: $x \neq y$, $x \nleq y$ or $A \not\subseteq B$ (*not* is denoted by a line "striking out" the relation symbol).

Let $A = \{1, 2, 3, 4\}$ and $B = \{0, 2, 4, 6\}$. Let us define the relation $R \subseteq A \times B$ as

$$R = \{(a, b) \in A \times B \mid a < b\}.$$

We have that

$$R = \{(1, 2), (1, 4), (1, 6), (2, 4), (2, 6), (4, 6)\}.$$

We can visualize any binary relation $R \subseteq A \times B$ by drawing the elements of A and drawing a line between an element a and an element b if $a\,R\,b$. In Figure 1 is depicted the above relation R.

A **directed graph** is an ordered pair $G = (V, E)$ where

- V is a set whose elements are called **vertices** (or **nodes** or **points**)

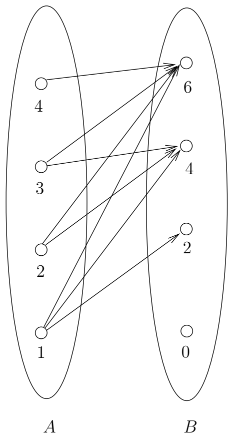


Figure 1: A relation R from A to B

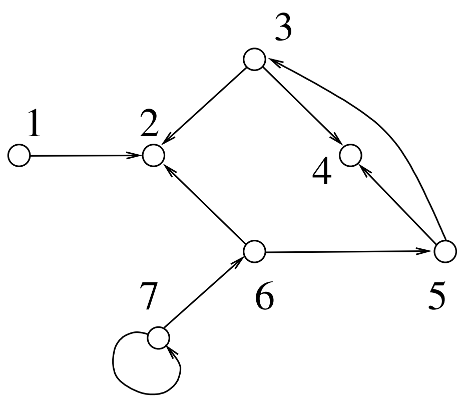


Figure 2: A directed graph

- E is a set of ordered pairs of vertices, called **edges** (or **arrows** or **lines**).

It is now clear that a binary relation R over a set A corresponds to a directed graph $G = (A, R)$, and each directed graph $G = (V, E)$ means that E is a binary relation on E.

In Figure 2 is given an example of directed graph. Using binary relations and graphs, it is possible to present all sorts of information. For instance, the set of vertices V may consist of web sites and if there is a link from site s to site t, then $(s, t) \in E$, that is, there is an arrow from s to t. **Graph theory** (`https://en.wikipedia.org/wiki/Graph_theory` studies properties of graphs.

Since relations are *sets*, we can construct new relations from existing ones by using set-theoretical operations. If R and S are relations from A to B, then their union $R \cup S$ and intersection $R \cap S$ are also relations from A to B. Similarly, R's complement $R^c = A \times B \setminus R$ is such a relation.

The **inverse relation** of a binary relation is the relation that occurs when the order of the elements is switched in the relation. For example, the inverse of the relation "child of" is the relation "parent of". In formal terms, if X and Y are sets and R is a relation from A to B, then R^{-1} is a relation from B to A defined so that $(y, x) \in R^{-1}$ whenever $(x, y) \in R$.

If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are two binary relations, then their **composition** $R \circ S$ is the relation

$$R \circ S = \{(x, z) \in X \times Z \mid \text{there is } y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}.$$

Example 6. The words *uncle* and *aunt* indicate a compound relation: for a person to be an uncle, he must be a brother of a parent. The relation of Uncle (denoted here by U) is the composition of relations "is a brother of" (denoted by B) and "is a parent of" (denoted P), that is,

$$U = B \circ P.$$

Similarly, aunt is a sister of a parent.

Lemma 7. *Let X, Y, and Z be sets. If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are two binary relations, then*

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}.$$

Proof. We prove that the "sets" $(R \circ S)^{-1}$ and $S^{-1} \circ R^{-1}$ are the same, that is, $(R \circ S)^{-1} \subseteq S^{-1} \circ R^{-1}$ and $(R \circ S)^{-1} \supseteq S^{-1} \circ R^{-1}$.

Note that $R \circ S$ is a relation from X to Z. Therefore, $(R \circ S)^{-1}$ is a relation from Z to X. Similarly, R^{-1} is a relation from Y to X and S^{-1} is a relation from Z to Y. This means that $S^{-1} \circ R^{-1}$ is a relation from Z to X.

Let $x \in Z$ and $y \in X$. If $(x, y) \in (R \circ S)^{-1}$, then $(y, x) \in R \circ S$. This means that there is $z \in Y$ such that $(y, z) \in R$ and $(z, x) \in S$. By the definition of the inverse relation, this means that $(z, y) \in R^{-1}$ and $(x, z) \in S^{-1}$. We have that there is $z \in Y$ such that $(x, z) \in S^{-1}$ and $(z, y) \in R^{-1}$. This means $(x, y) \in S^{-1} \circ R^{-1}$. We have now proved $(R \circ S)^{-1} \subseteq S^{-1} \circ R^{-1}$.

On the other hand, let $a \in Z$ and $b \in X$. If $(a, b) \in S^{-1} \circ R^{-1}$, then there is $c \in Y$ such that $(a, c) \in S^{-1}$ and $(c, b) \in R^{-1}$. We obtain that $(b, c) \in R$ and $(c, a) \in S$. Hence, $(b, a) \in R \circ S$ and $(a, b) \in (R \circ S)^{-1}$. We have now proved the converse inclusion $S^{-1} \circ R^{-1} \subseteq (R \circ S)^{-1}$.

Therefore, $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. $\square$

We end this section by considering generalizations of binary relations. Let $n \geq 1$ be an integer. An **n-ary relation** R over sets $A_1, \ldots, A_n$ is a subset of the Cartesian product $A_1 \times \cdots \times A_n$, that is,

$$R \subseteq A_1 \times A_2 \times \cdots \times A_n.$$

This means that an n-ary relation is a set consisting of n-vectors $(x_1, x_2, \ldots, x_n)$ such that x_i belongs to A_i for all $1 \leq i \leq n$, that is, the first component x_1 of the vector $(x_1, x_2, \ldots, x_n)$ belongs to the set A_1, the second component x_2 belong to A_2, and so on. Finally, the nth component x_n of the vector $(x_1, x_2, \ldots, x_n)$ belongs to the set A_n.

Example 8. Let $n = 3$ and $A_1 = \mathbb{Z}$, $A_2 = \mathbb{Z}$, and $A_3 = \mathbb{Z}$. Define a 3-ary relation R such that

$$(a, b, c) \in R \text{ if there is an integer } k \text{ such that } b = a + k \text{ and } c = a + 2k.$$

Now for example $(4, 3, 2)$ belongs to R, because $k = -1$ is a suitable integer.

In fact, we can solve $k = b - a$ and put this to the other equation. Therefore, we have the condition between the numbers a, b, and c:

$$c = a + 2(b - a) = a + 2b - 2a = 2b - a.$$

If $a = 62$ and $b = 99$, then $c = 2 \cdot 99 - 62 = 198 - 62 = 136$. This means that $(62, 99, 136) \in R$.

The relationship R defines here a relationship between three integers, but not all three integers are in this relation. For instance, $(3, 3, 3) \in R$, but $(3, 3, 4) \notin R$.

If $R \subseteq A_1 \times A_2 \times \cdots \times A_n$, the sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation and n is its **degree**.

Example 9. Tables with n columns can be viewed as an n-ary relation, where rows correspond n-vectors and columns correspond domains.

For instance, consider the relation $R = \textsc{DepartingFlights}$ which describes a small amount of flights departing from London Heathrow airport in one evening. The degree of R is $n = 6$ and the domains of the relation are:

- $A_1 = \textsc{scheduled}$ consists of all possible times
- $A_2 = \textsc{flight}$ consists of all possible flight numbers
- $A_3 = \textsc{departing\_to}$ consists of airports
- $A_4 = \textsc{airline}$ consists of airlines
- $A_5 = \textsc{terminal}$ consists of the terminas in Heathrow
- $A_6 = \textsc{status}$ consists of the possible statuses of the the flights

Each row (6-vector) represents on departing flight.

SCHEDULED	FLIGHT	DEPARTING_TO	AIRLINE	TERMINAL	STATUS
21:05	EI179	Dublin	Aer Lingus	T2	taxied
21:25	BA057	Johannesburg	British Airways	T5	boarding
21:25	FI455	Reykjavik	Icelandair	T2	taxied
21:55	QR016	Doha	Qatar Airways	T5	flight closing
22:05	GF006	Bahrain	Gulf Air	T2	gate open
22:10	VS449	Johannesburg	Virgin Atlantic	T3	on time

In symbols:

$$\textsc{DepartingFlights} \subseteq$$
$$\textsc{scheduled} \times \textsc{flight} \times \textsc{departing\_to} \times \textsc{airline} \times \textsc{terminal} \times \textsc{status}.$$

The so-called **relational databases** are based on such a relational model. Columns are also called *attributes*. Many relational database systems are using the SQL (Structured Query Language) for querying and maintaining the database.

It is also possible to have a 1-ary relation on X, commonly known as a **unary relation**. Then a unary relation R on X is just a subset of X. Unary relations are also called **predicates**. A unary relation (or predicate) P expresses a property of objects, and $P(x)$ holds if x has the property P. We have already used predicates when we considered set-builders:

$$A = \{x \in X \mid P(x)\}.$$

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