Logic circuits and functions

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Control logic

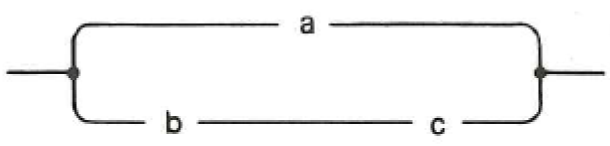
- In modern machines, the vast majority of control & information transfer processes are done in digital fashion
- The control is practically implemented via logic gates
- These gates are basically propositions connected by connectives
  - We are already familiar with most of these
- Gates are constructed from simple electrical parts: diodes, transistors and relays
- Control is based on Boolean algebra

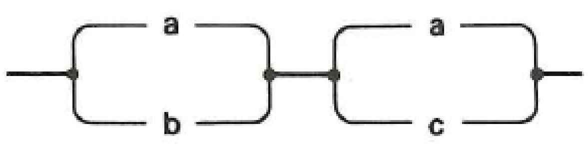
Boolean algebra

- In Boolean algebra, variables have only a truth value (1/0, or TRUE / FALSE, respectively)
- The basic Boolean functions are the same as the connectives in propositional logic:
  - Conjunction $(a \wedge b)$
  - Disjunction $(a \vee b)$
  - Negation $(\neg a)$
- In Boolean algebra, following simplified shorthand notation is often used:
  - Conjunction ab (AND = Boolean multiplication)
  - Disjunction $a + b$ (OR = Boolean addition)
  - Negation $\bar{a}$ or alternatively a'

Boolean algebra

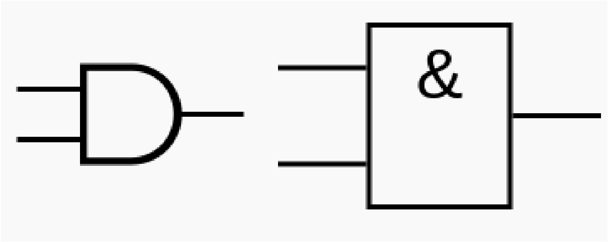
- In Boolean algebra, it is important to notice the following peculiarities:
  - 1 + 1 = 1 (OR)
  - a + bc = (a + b)(a + c)
- Both of these seem to be against the rules of “regular” algebra, but these rules can be shown to be correct by
  - Constructing a truth table
  - Sketching the graphs of these circuits



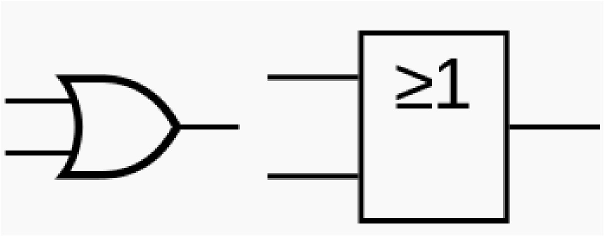


Logic gates

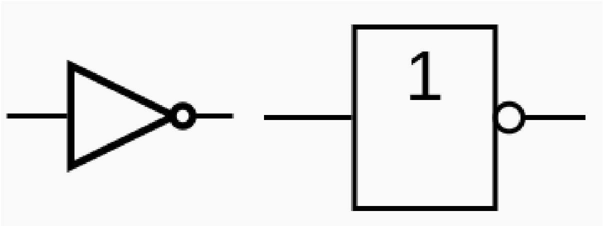
- Logic gates have their own drawing symbols
  - Left: American version (more common)
  - Right: European version
  - AND



  - OR



  - NOT

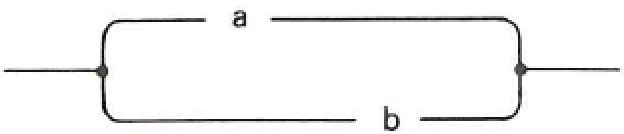


Simplified circuit diagram

- A logic circuit can be drawn also without “fancy” symbols using just a simple basic idea:
  - Draw all possible situations where the truth value is 1 – so, a signal can travel from left to right
  - Current values of variables are indicated by variable letters and their negations as in shorthand notation
  - AND = series

$F = ab$

  - OR = parallel



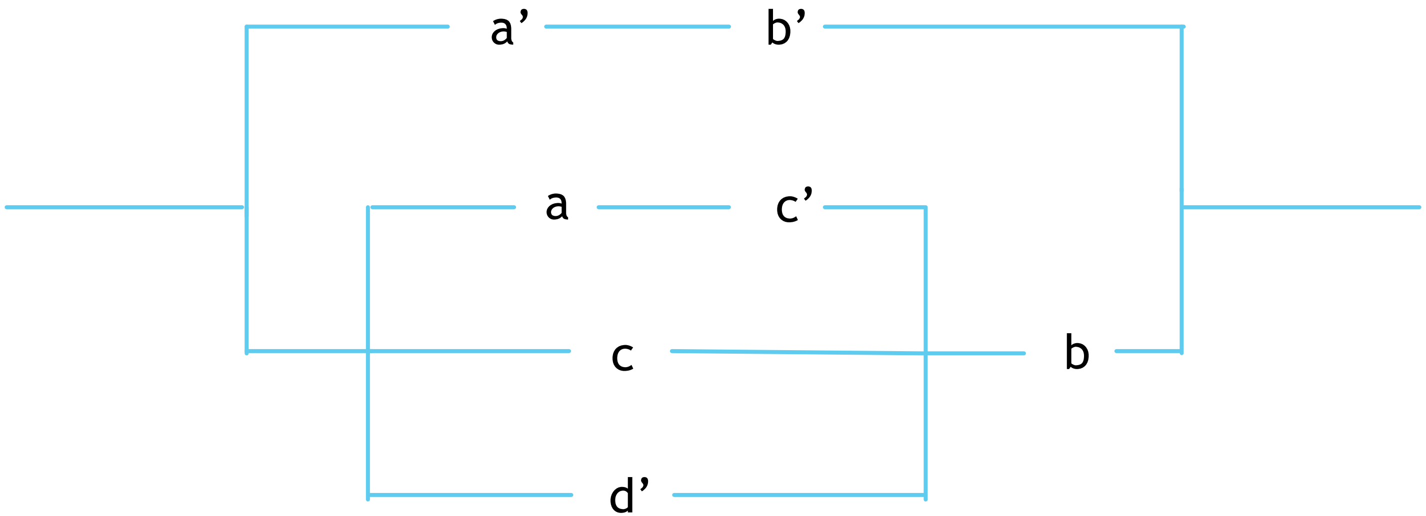
$F = a + b$

- NOTE: This notation only allows negations of sole variables, so we might have to use De Morgan’s laws

$$\overline{ab} \Leftrightarrow \bar{a} + \bar{b} \qquad \overline{a + b} \Leftrightarrow \bar{a}\bar{b}$$

Simplified circuit diagram

- Simplified circuit diagram is often easier to understand for people who are not familiar with electronics, computer science or advanced mathematics
- Example:



  - Logic function: $a'b' + (ac' + c + d')b$
  - Expanded SOP-form: $a'b' + abc' + bc + bd'$

Forms of logic function

- The switching logic can be presented in three forms:
  - Truth table (we’ve done these already)
  - Logic function in Boolean algebra
  - Circuit diagram (official or simplified)

- The logic function can be constructed in two different normal forms:
  - Disjunctive normal form DNF (Sum Of Products, SOP)
  - Conjunctive normal form CNF (Product Of Sums, POS)
- The difference is significant

Sum of Products (SOP)

- In SOP-form the logic function is constructed of products of which each of them gives a truth value of 1
- A product term which includes all the variables is called a minterm
- One minterm corresponds to exactly one combination in the truth table; for this combination the minterm gets a truth value of 1 and all other combinations cause the minterm go to 0
- Summing up all the minterms produces a standard SOP form for the circuit
- In computer science this form is called disjunctive normal form (DNF)
- A SOP-circuit always has two levels:
  - First all AND operations
  - In the end all OR operations (between minterms)

Product of Sums (POS)

- In POS-form the logic function is constructed of sum terms of which each of them gives a truth value of 0
- A sum term which includes all the variables is called a maxterm
- NOTE! Now the negations in maxterms are inverted, because the truth value must be zero!
  - For example, if in truth table the truth value of A is 1, we have to write the negation of A in the sum term; see example on next slide)
- Multiplying all the maxterms produces a standard POS form for the circuit
- In computer science this form is called conjunctive normal form (CNF)
- A POS-circuit always has two levels, too:
  - First all OR operations
  - In the end all AND operations (between maxterms)

Minterms and maxterms

- Example of minterms and maxterms in a three-variable circuit case:

X Y Z	Minterms		Maxterms	
	Product	Symbol	Sum	Symbol
0 0 0	$\bar{X}\bar{Y}\bar{Z}$	m0	$X + Y + Z$	M0
0 0 1	$\bar{X}\bar{Y}Z$	m1	$X + Y + \bar{Z}$	M1
0 1 0	$\bar{X}Y\bar{Z}$	m2	$X + \bar{Y} + Z$	M2
0 1 1	$\bar{X}YZ$	m3	$X + \bar{Y} + \bar{Z}$	M3
1 0 0	$X\bar{Y}\bar{Z}$	m4	$\bar{X} + Y + Z$	M4
1 0 1	$X\bar{Y}Z$	m5	$\bar{X} + Y + \bar{Z}$	M5
1 1 0	$XY\bar{Z}$	m6	$\bar{X} + \bar{Y} + Z$	M6
1 1 1	XYZ	m7	$\bar{X} + \bar{Y} + \bar{Z}$	M7

Example 1

- Formulate a logic function that fulfills the following truth table using
  - a) Minterms (SOP-form)
  - b) Maxterms (POS-form)

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

a) SOP-form

A	B	C	F	Minterm
0	0	0	0	
0	0	1	0	
0	1	0	1	$\bar{A}B\bar{C}$
0	1	1	0	
1	0	0	1	$A\bar{B}\bar{C}$
1	0	1	1	$A\bar{B}C$
1	1	0	0	
1	1	1	1	ABC

$$\boldsymbol{F = \overline{A}B\overline{C} + A\overline{B}\overline{C} + A\overline{B}C + ABC}$$

b) POS-form

A	B	C	F	Maxterm
0	0	0	0	$A + B + C$
0	0	1	0	$A + B + \bar{C}$
0	1	0	1	
0	1	1	0	$A + \bar{B} + \bar{C}$
1	0	0	1	
1	0	1	1	
1	1	0	0	$\bar{A} + \bar{B} + C$
1	1	1	1	

$$\boldsymbol{F = (A + B + C)(A + B + \bar{C})(A + \bar{B} + \bar{C})(\bar{A} + \bar{B} + C)}$$

Simplification of a logic function

- Logic functions presented in their normal forms do work, but they are often overtly complicated
- Usually it’s possible to simplify the functions
- Minimization of a logic function is, in general, an NP-hard problem
  - Possible, but takes A LOT of time
- Minimization can still be done to some extent in reasonable time
- Tools that are used for this include:
  - Boolean algebra (simplification)
  - Karnaugh map (“K-map”)
  - Certain computer software

Formulae of Boolean algebra

One-variable theorems:

$X + 0 = X$

$X + 1 = 1$

$X + X = X$

$X + \overline{X} = 1$

$\overline{\overline{X}} = X$

$X \cdot 1 = X$

$X \cdot 0 = 0$

$X \cdot X = X$

$X \cdot \overline{X} = 0$

Several-variable theorems:

$X + Y = Y + X$

$X + (Y + Z) = (X + Y) + Z$

$X(Y + Z) = X\,Y + X\,Z$

$X\,Y = Y\,X$

$X(Y\,Z) = (X\,Y)Z$

$X + Y\,Z = (X + Y)(X + Z)$

De Morgan's laws

$\overline{X + Y} = \overline{X} \cdot \overline{Y}$

$\overline{X \cdot Y} = \overline{X} + \overline{Y}$

Consensus theorem

- In addition to previously mentioned formulae, so called "consensus theorem" allows us to remove terms which are not needed:

$$AB + \overline{A}C + BC = AB + \overline{A}C$$

- Term BC is unnecessary. Why? Let's examine using set theory:

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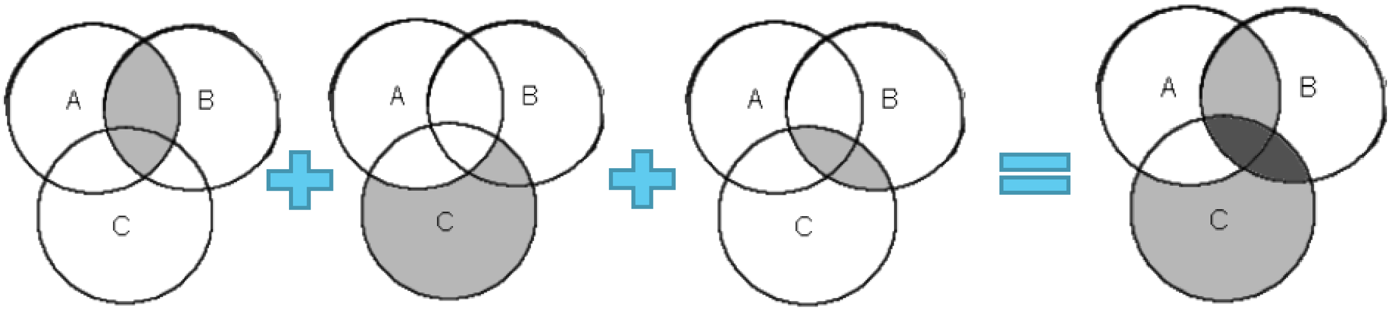

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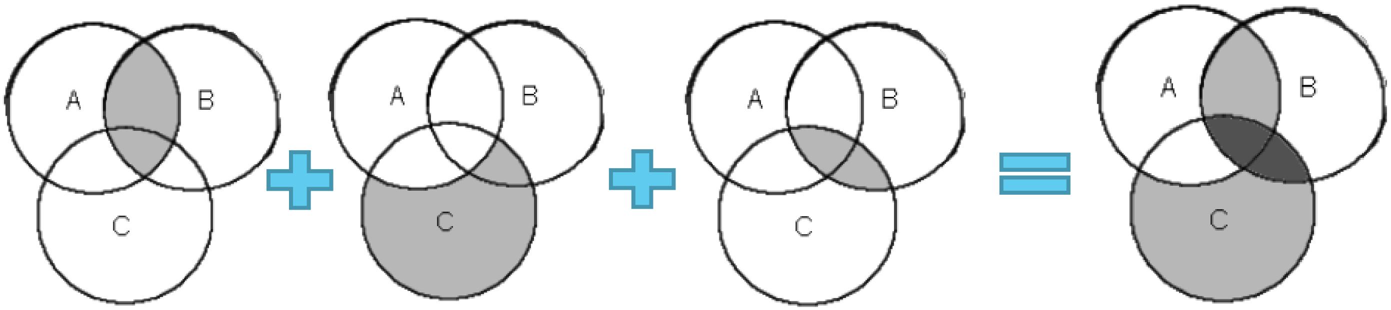


- Area BC is colored twice; this means that it is already included in our chosen set. Therefore the term BC can left out.

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) SOP-form b) POS-form.

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) SOP-form

$$F = \overline{A}B\overline{C} + A\overline{B}\overline{C} + A\overline{B}C + ABC$$

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) SOP-form

$$F = \overline{A}B\overline{C} + A\overline{B}\overline{C} + A\overline{B}C + ABC$$

Duplication

$$= \overline{A}B\overline{C} + (A\overline{B}\overline{C} + A\overline{B}C) + (A\overline{B}C + ABC)$$

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) SOP-form

$$F = \overline{A}B\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C + ABC$$

Duplication

$$= \overline{A}B\overline{C} + (A\overline{B}\,\overline{C} + A\overline{B}C) + (A\overline{B}C + ABC)$$

$$= \overline{A}B\overline{C} + A\overline{B}(\overline{C} + C) + AC(\overline{B} + B)$$

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) SOP-form

$$F = \overline{A}B\overline{C} + A\overline{B}\overline{C} + A\overline{B}C + ABC$$

Duplication

$$= \overline{A}B\overline{C} + (A\overline{B}\overline{C} + A\overline{B}C) + (A\overline{B}C + ABC)$$

$$= \overline{A}B\overline{C} + A\overline{B}(\overline{C} + C) + AC(\overline{B} + B)$$

= 1

$$= \overline{A}B\overline{C} + A\overline{B} + AC$$

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) SOP-form

$$F = \bar{A}B\bar{C} + A\bar{B}\bar{C} + A\bar{B}C + ABC$$

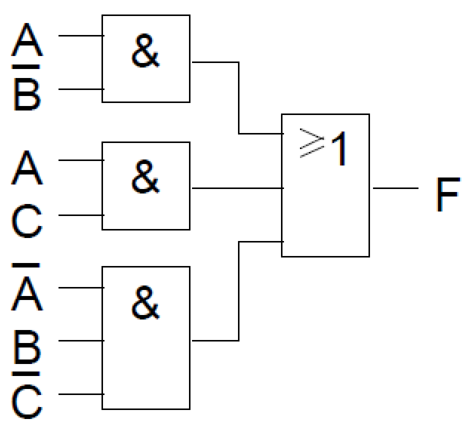
Duplication

$$= \bar{A}B\bar{C} + (A\bar{B}\bar{C} + A\bar{B}C) + (A\bar{B}C + ABC)$$

$$= \bar{A}B\bar{C} + A\bar{B}(\bar{C} + C) + AC(\bar{B} + B)$$

= 1

$$= \bar{A}B\bar{C} + A\bar{B} + AC$$



Simplified circuit:
4 gates, 2+2+3+3
= 10 inputs

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) POS-form

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- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) POS-form

$$\boldsymbol{F} = (\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C})(\boldsymbol{A}+\boldsymbol{B}+\overline{\boldsymbol{C}})(\boldsymbol{A}+\overline{\boldsymbol{B}}+\overline{\boldsymbol{C}})(\overline{\boldsymbol{A}}+\overline{\boldsymbol{B}}+\boldsymbol{C})$$

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) POS-form

Duplication

$$F = (A + B + C)(A + B + \overline{C})(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

$$= (A + B + C)(A + B + \overline{C})(A + B + \overline{C})(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) POS-form

Duplication

$$F = (A + B + C)(A + B + \overline{C})(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

$$= (A + B + C)(A + B + \overline{C})(A + B + \overline{C})(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

Consensus theorem

Consensus theorem

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) POS-form

Duplication

$$F = (A + B + C)(A + B + \overline{C})(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

$$= (A + B + C)(A + B + \overline{C})(A + B + \overline{C})(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

Consensus theorem   Consensus theorem

$$= (A + B)(A + \overline{C})(\overline{A} + \overline{B} + C)$$

Example 1 (continued)

- Simplify the logic functions of example 1 and present the circuit diagram in
  - a) POS-form

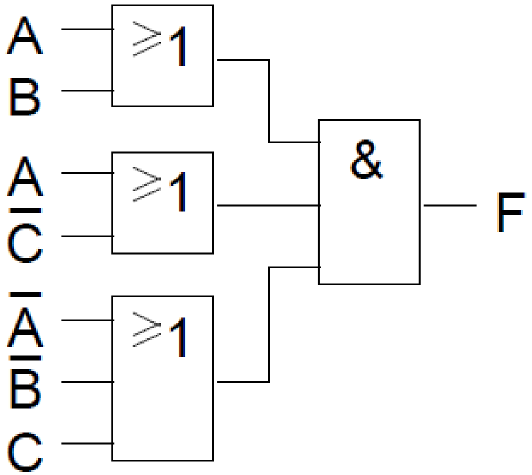
Duplication

$$F = (A + B + C)(A + B + \overline{C})(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

$$= (A + B + C)(A + B + \overline{C})(A + B + \overline{C})(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

Consensus theorem   Consensus theorem

$$= (A + B)(A + \overline{C})(\overline{A} + \overline{B} + C)$$



Simplified circuit:
4 gates, 2+2+3+3
= 10 inputs

SOP or POS?

- Which normal form is easier, then?
- SOP is often considered “easier for brains”
  - No need to invert truth values
  - Easier to simplify (Boolean) algebraically
- The end result might be simpler in SOP- or POS-form
  - No way to tell unless you have a “trained eye”
- One heuristic approach:
  - In truth table, F = 1 for minterms and F = 0 for maxterms
  - If there are less minterms, SOP form gives a simpler starting point
  - If there are less maxterms, POS form gives a simpler starting point
  - …but situation can change due to simplification!

Example 2

- Formulate a logic function that corresponds to the truth table on the right using both SOP and POS forms.
- Simplify your results.

A	B	C	D	F
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	0
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

Example 2

- Formulate a logic function that corresponds to the truth table on the right using both SOP and POS forms.
- Simplify your results.

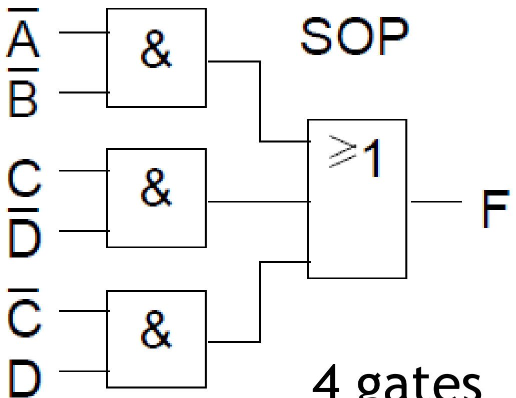
10 minterms, 6 maxterms. POS might be a smarter solution?

A	B	C	D	F
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	0
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

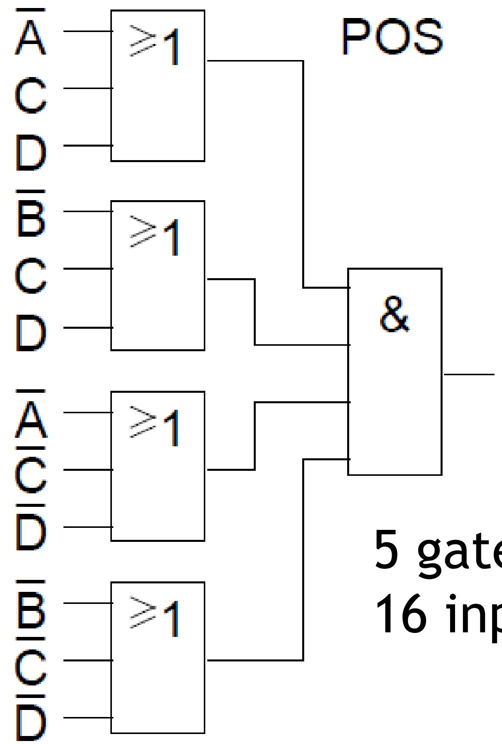
Example 2

- When simplified, the situation is reversed:

$F = \overline{A}\,\overline{B} + C\,\overline{D} + \overline{C}\,D$ (SOP)



4 gates
9 inputs



5 gates
16 inputs

$F = (\overline{A} + C + D)(\overline{B} + C + D)(\overline{A} + \overline{C} + \overline{D})(\overline{B} + \overline{C} + \overline{D})$ (POS)

Thank you!