Propositional logic

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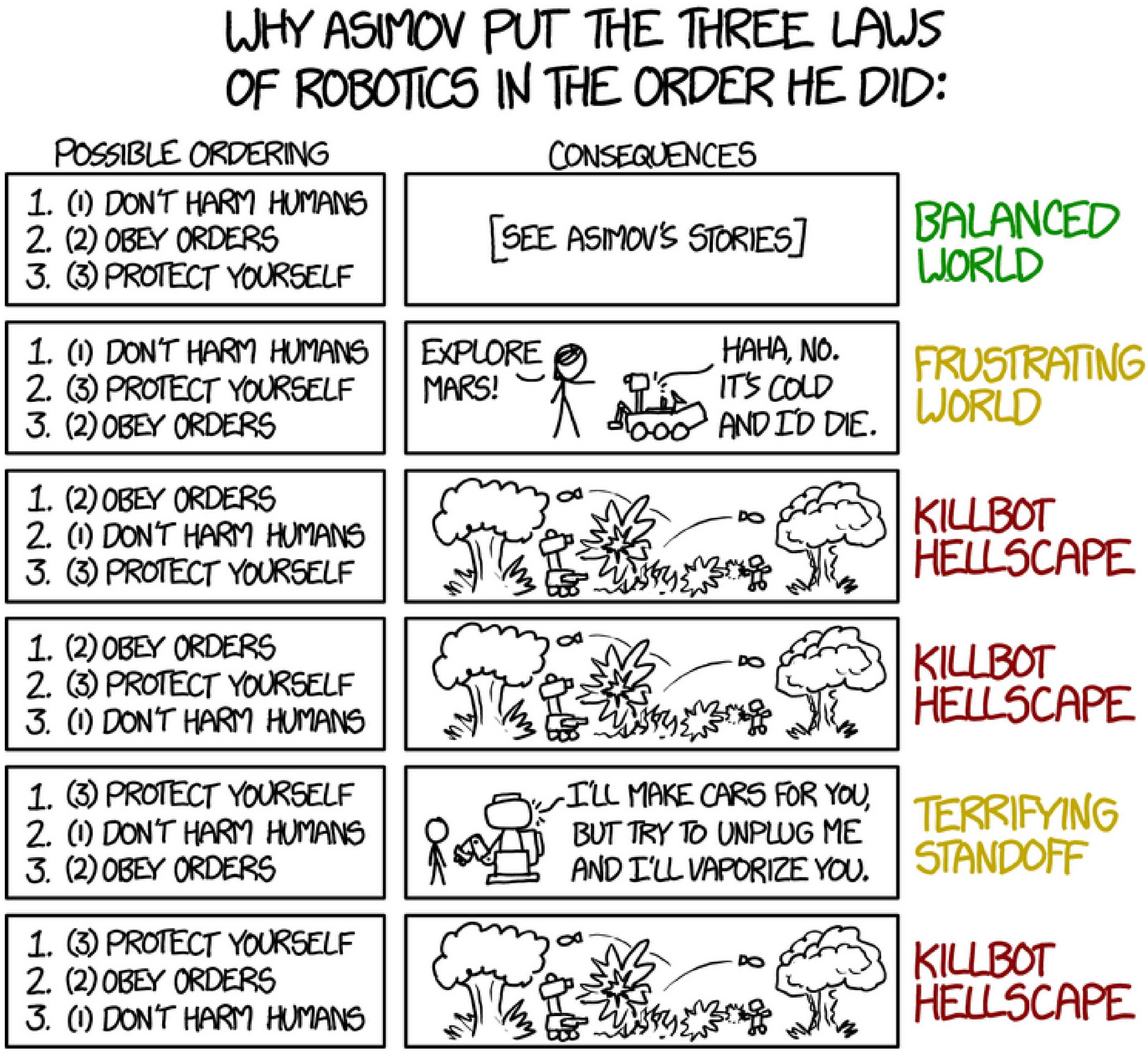
Logic

- Logic is a branch of science which emphasizes deduction and deduction processes
- Traditionally considered to belong in the field of philosophy, but plays a significant role also in mathematics
- Important in technical applications – for example process logic of a machine:
  - What does the machine do and in which order?
  - What function is prioritized in case of multiple inputs?
- The more we outsource decisions to artificial intelligence (AI), the more important it is to define the working logic by means of discrete mathematics
- Ethical issues need to be addressed, too
  - For example, self-driving cars


WHY ASIMOV PUT THE THREE LAWS OF ROBOTICS IN THE ORDER HE DID:
POSSIBLE ORDERING
CONSEQUENCES
1. (1) DON'T HARM HUMANS
2. (2) OBEY ORDERS
3. (3) PROTECT YOURSELF
[SEE ASIMOV'S STORIES]
BALANCED WORLD
1. (1) DON'T HARM HUMANS
2. (3) PROTECT YOURSELF
3. (2) OBEY ORDERS
EXPLORE MARS!
HAHA, NO. IT'S COLD AND I'D DIE.
FRUSTRATING WORLD
1. (2) OBEY ORDERS
2. (1) DON'T HARM HUMANS
3. (3) PROTECT YOURSELF
KILLBOT HELLSCAPE
1. (2) OBEY ORDERS
2. (3) PROTECT YOURSELF
3. (1) DON'T HARM HUMANS
KILLBOT HELLSCAPE
1. (3) PROTECT YOURSELF
2. (1) DON'T HARM HUMANS
3. (2) OBEY ORDERS
I'LL MAKE CARS FOR YOU, BUT TRY TO UNPLUG ME AND I'LL VAPORIZE YOU.
TERRIFYING STANDOFF
1. (3) PROTECT YOURSELF
2. (2) OBEY ORDERS
3. (1) DON'T HARM HUMANS
KILLBOT HELLSCAPE

Proposition

- Proposition is an expression that includes a claim
- Claim can be true (truth value = 1) or false (truth value = 0)
- The truth value of a proposition is fixed – there's no variable inside the proposition; no ifs or buts
- Logic that uses only propositions is called propositional logic or 0^{th}-order logic
  - We'll advance to first-order logic in the next lecture; it's probably easier to spot the difference, then
- Examples of atomic propositions:
  - p = "it's snowing outside"
  - q = "a leopard is a feline animal"
  - r = "Robert makes it to train in time"

Connectives

- In propositional logic we examine the properties of propositions formed from atomic propositions
- New propositions are formed from atomic propositions by using different connectives
- There are a lot of these connectives (very advanced ones too), but in principle, they all can be broken down to a couple of basic connectives
  - Negation ($\neg p$)
  - Conjunction ($p \wedge q$)
  - Disjunction ($p \vee q$)
  - Implication ($p \Rightarrow q$)
  - Equivalence ($p \Leftrightarrow q$)
- Let’s examine these connectives using truth tables

Negation ($\neg p$)

- “Not p” (compare in statistical mathematics: complement)
- Equivalents in so called natural language include
  - “No”
  - “It is not true, that…”
  - “We can’t say that…”
- The definition is rather self-explanatory
- So is the truth table:

p	$\neg p$
0	1
1	0

Conjunction ($p \wedge q$)

- "p AND q" (compare in statistical mathematics: intersection)
- Equivalents in natural language:
  - "and"
  - "both … and …"
- Also rather self-explanatory concept
- Truth table shows that the statement is true only when p = 1 and q = 1

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Disjunction ($p \vee q$)

- “p OR q” (compare in statistical mathematics: union)
- Equivalents in natural language:
  - “or” / “either… or …”
- Interpretation problem in conversion from natural language to logic: how about if both are true?
  - In logic, disjunction is interpreted in such a way that both can be true at the same time
  - In natural language the or can be meant as exclusive (for example: “coffee or tea for dessert” = pick one but not both)
- Truth table is as follows:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Implication ($p \Rightarrow q$)

- "if p, then q"
- Equivalents in natural language include also
  - "when …, …"
  - In natural language people often use implication when they actually mean equivalence; be careful with conversion!)
- NOTE! False only when p is true and q is not
  - This might seem weird, but the definition is logical (pun intended): if the statement p is false, we can't prove the implication to be false
- Therefore the truth table looks like this:

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Equivalence ($p \Leftrightarrow q$)

- "p if and only if q" (or "p iff q" for short)
- As the definition says, true only then when the truth values of p and q are the same
- In logic, the concept of equivalence is clear, but unambiguous interpretation of expressions in natural language is sometimes hard
  - Context is often needed
- Truth table:

p	q	$p \Leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Compound propositions

- Using connectives, we can formulate compound propositions from atomic propositions
- These compound propositions can be quite long strings
- It's usually favorable to use brackets in order to ensure accurate interpretation
- In addition, connectives are agreed to be applied in following order (compare to order of operations in "regular" mathematics):
  - 1. Negations
  - 2. Conjunctions & disjunctions
  - 3. Implications and equivalences
- This order somewhat reduces the need for brackets
- The truth value of the compound proposition can be examined using a truth table
  - It's often wise to compose the truth table phase by phase

Example 1

- Examine the truth values of the proposition "if I go to a kiosk or store, I'll go to a store and buy beer"

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- Construct a truth table. Because there are three atomic propositions and each can be either 1 (true) or 0 (false), the number of rows in a truth table will be $2^3 = 8$
- It’s a good idea to define truth values for the following propositions as mid-results
  - $p \vee r$ (“I go to a kiosk or store”)
  - $q \wedge r$ (“I go to a store and buy beer”)

Example 1

- Truth table will then look like this:

p	q	r	$p \vee r$	$q \wedge r$	$p \vee r \Rightarrow q \wedge r$
0	0	0	0	0	1
0	0	1	1	0	0
0	1	0	0	0	1
0	1	1	1	1	1
1	0	0	1	0	0
1	0	1	1	0	0
1	1	0	1	0	0
1	1	1	1	1	1

Tautologies

- If the newly formed proposition is always true, this proposition is called a tautology
- This means that the truth value of the proposition is always 1 – no matter what the truth values of its atomic propositions are
- A proposition can be proven to be a tautology by constructing a truth table
  - If the end column consists of only 1s, it's a tautology
  - The previous example wasn't, obviously
- Let's now examine a couple of propositions that have been proven to be tautologies
  - Tautologies form logical rules

Example 2

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- Let's proceed in peace, one connective at a time:
  - Start with atomic propositions p and q
  - Find out truth value of $p \Rightarrow q$
  - Next, truth value of $(p \Rightarrow q) \wedge p$
  - After this, final truth value for the whole proposition

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p	q	$p \Rightarrow q$	$(p \Rightarrow q) \land p$	$(p \Rightarrow q) \land p \Rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

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0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

- We notice that the proposition is a tautology.

Commonly known tautologies

Tautology	Name
$p \Leftrightarrow p$	Law of identity
$\neg(p \wedge \neg p)$	Law of non-contradiction
$p \vee \neg p$	Law of excluded middle
$\neg\neg p \Leftrightarrow p$	Law of double negation
$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$	De Morgan’s Theorem 1
$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$	De Morgan’s Theorem 2
$p \Rightarrow q \Leftrightarrow \neg p \vee q$	Definition of implication 1
$p \Rightarrow q \Leftrightarrow \neg(p \wedge \neg q)$	Definition of implication 2

Commonly known tautologies

Tautology	Name
$p \wedge p \Leftrightarrow p$	Idempotent law 1
$p \vee p \Leftrightarrow p$	Idempotent law 2
$p \wedge q \Leftrightarrow q \wedge p$	Commutation law 1
$p \vee q \Leftrightarrow q \vee p$	Commutation law 2
$(p \Leftrightarrow q) \Leftrightarrow (q \Leftrightarrow p)$	Commutation law 3
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	Association law 1
$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	Association law 2
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distribution law 1
$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	Distribution law 2

How to use tautologies?

- According to rules of logic, we can always replace any part of a proposition by a logically equivalent part
  - “Logically equivalent” = has identical truth table
- Hence, we can use tautologies in order to simplify propositions
- Notice that logically equivalent propositions may have different nuances in natural language
  - “We have to balance the budget, but raising taxes is not an option” sounds better than “We have to balance the budget, so we cut the public services and social benefits”
  - “I can’t say that we played well” = “we played poorly”

Example 3: Simplify

$$\neg(p \vee q \Rightarrow \neg p \wedge \neg q)$$

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$\neg(p \vee q \Rightarrow \neg p \wedge \neg q)$ Definition of implication 2

$\equiv \neg\neg((p \vee q) \wedge \neg(\neg p \wedge \neg q))$

Example 3: Simplify

$$\neg(p \vee q \Rightarrow \neg p \wedge \neg q)$$ Definition of implication 2

$$\equiv \neg\neg\big((p \vee q) \wedge \neg(\neg p \wedge \neg q)\big)$$ Double negation & De Morgan 2

$$\equiv (p \vee q) \wedge (\neg\neg p \vee \neg\neg q)$$

Example 3: Simplify

$\neg(p \vee q \Rightarrow \neg p \wedge \neg q)$	Definition of implication 2
$\equiv \neg\neg((p \vee q) \wedge \neg(\neg p \wedge \neg q))$	Double negation & De Morgan 2
$\equiv (p \vee q) \wedge (\neg\neg p \vee \neg\neg q)$	Double negation
$\equiv (p \vee q) \wedge (p \vee q)$	

Example 3: Simplify

$\neg(p \vee q \Rightarrow \neg p \wedge \neg q)$	Definition of implication 2
$\equiv \neg\neg((p \vee q) \wedge \neg(\neg p \wedge \neg q))$	Double negation & De Morgan 2
$\equiv (p \vee q) \wedge (\neg\neg p \vee \neg\neg q)$	Double negation
$\equiv (p \vee q) \wedge (p \vee q)$	Idempotent law
$\equiv p \vee q$	

Example 3: Simplify

$$\neg(p \vee q \Rightarrow \neg p \wedge \neg q)$$ Definition of implication 2

$$\equiv \neg\neg((p \vee q) \wedge \neg(\neg p \wedge \neg q))$$ Double negation & De Morgan 2

$$\equiv (p \vee q) \wedge (\neg\neg p \vee \neg\neg q)$$ Double negation

$$\equiv (p \vee q) \wedge (p \vee q)$$ Idempotent law

$$\equiv p \vee q$$

The examined proposition is therefore equivalent to the disjunction of p and q. We could even say that…

$$\neg(p \vee q \Rightarrow \neg p \wedge \neg q) \Leftrightarrow p \vee q$$

…is a tautology.

Thank you!