Induction and recursion

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Induction

- In science, inductive reasoning means making generalizations based on single (but numerous) observations
  - Opposite to deductive reasoning, where general principles are used as a starting point
- Based on this definition, it seems like inductive method is no good for proving anything mathematically:
  - Problem: is equation $f(x,y,z) = g$ always true?
  - Try 100 (x,y,z)-combinations and notice that the function $f(x,y,z)$ returns a constant value g every time
  - This is not sufficient proof that the equation is always true, because in order to claim that, we'd have to try every possible value (which is impossible, if the domain of x,y,z is infinite)

Proof by induction

- There still exists a mathematically sound method, which is called (due to its nature) proof by induction or mathematical induction
  - This is a set phrase, even though it’s not accurate
- Using proof by induction, we can prove several formulae linked to for example
  - Sequences, series and sums
  - Inequalities
  - Division problems

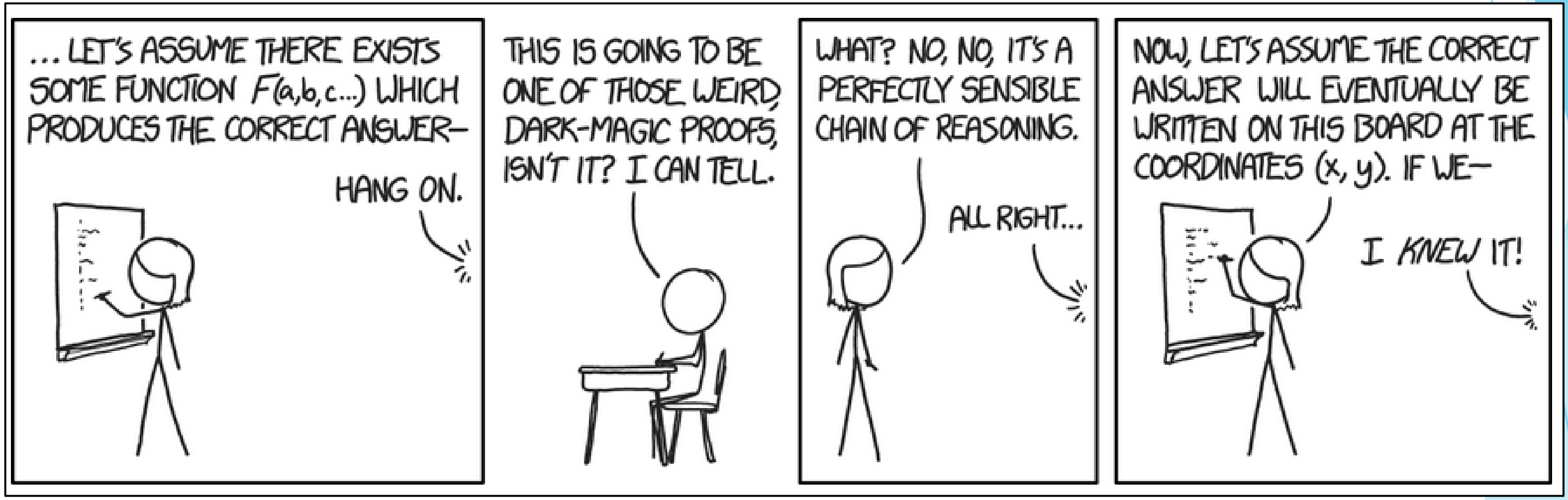
Proof by induction

- If the task is to prove that the claim p_n is true for all $n \in \mathbb{N}$, proof by induction process goes in the following way:
  - Basic step (BS): Prove, that p_0 is true
  - Induction hypothesis (IH): Assume, that p_n is true, when $n = k$
  - Induction step (IS): With the help of the IH, prove that p_n is true also when $n = k + 1$
  - Conclusion (C): If the previous step was successful, we make a conclusion that the original claim p_n is true for all $n \in \mathbb{N}$
- The process can be formalized also using predicate logic:

$$\frac{\begin{array}{l} p_0 \\ \forall k \in \mathbb{N}: (p_k \Rightarrow p_{k+1}) \end{array}}{\forall k \in \mathbb{N}: p_n}$$

Proof by induction

- Induction proofs often seem confusing, and it is common that students are not convinced of their power



- Let's practice via some examples!

Proof by induction: Example 1

- Let's start with an easy example: use the proof by induction in order to prove the following sum formula for consecutive integers

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- Let’s take the base step first:
  - The first term of the series is 1, so let’s check the validity of the claim when $n = 1$

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BS:

$$\color{red}{Left\ side = 1}$$

$$\color{green}{Right\ side = \frac{1 \cdot (1+1)}{2} = \frac{1 \cdot 2}{2} = \frac{2}{2} = 1}$$

  - Left side = right side, so base step is ok!

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- Next let's make the induction hypothesis:

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  - The left side of the claim can be simplified by substituting the IH in place of $1 + 2 + 3 + \cdots + k$:

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  - Let's simplify the right side:

$$\frac{(k+1)\big((k+1)+1\big)}{2} = \frac{(k+1)(k+2)}{2} = \frac{k^2 + 2k + k + 2}{2} = \frac{k^2 + 3k + 2}{2}$$

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  - Left side = right side, so the claim is true!

- Conclusion: formula is correct

Proof by induction: Example 2

- Next, let's try to use proof by induction for inequalities. Let's prove that claim $n! > 3^n$ is true for all integers $n \geq 7$:

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- Basic step: Check the first value $n = 7$

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$$\underbrace{n!}_{\text{Left side}} > \underbrace{3^n}_{\text{Right side}}$$

- Basic step: Check the first value $n = 7$

BS:

$$Left\ side = 7! = 5040$$

$$Right\ side = 3^7 = 2187$$

- 5040 > 2187, so left side > right side as it was claimed. Basic step is ok!

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  - We can simplify this claim to form

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$$if\ a > c\ and\ b > d \Rightarrow ab > cd$$

  - With the help of IH we can say that because $k! > 3^k$, the IS claim is for sure true if also (k+1) > 3 is true. Because the domain of k was defined to be at least 7, this means that (k+1) > 3 always. Hence, the claim in IS is always true.

- Conclusion: the original inequality claim is correct!

Recursion

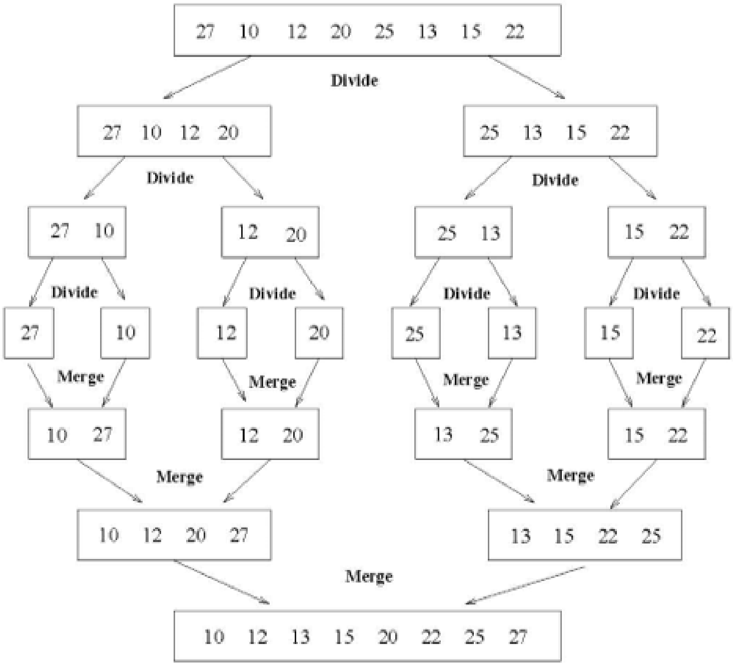
- A concept that is very closely linked to induction is recursion
- In induction, we progressed in a series all the way to *k* and took one step further from there
- In recursion we go in the other direction: we define a term of greater order number with the help of prior ones
  - Recursion formula defines how f_{n+1} is dependent of the former function values $f_n,\ f_{n-1},\ f_{n-2}, \dots$
- The order number of recursion formula tells how many prior values are needed* for this definition
  - If the next value is calculated solely based on the previous one, the recursion formula is of 1st order
  - If the next value is calculated based on, say, three previous ones, the recursion formula is of 3rd order

*Or, to be exact, how far back do we need to go. For example, $f_{n+1} = 3f_{n-1} + 2f_{n-8}$ would be a recursion formula of 9th order.

Recursion

- Before we've used recursive definition mostly in case of sequences and series
- In reality, many prediction models (especially time-series based ones) are defined in recursive fashion
  - Example: prediction model of electricity spot price

Recursion can be used also in programming, if we want to divide large problems into smaller subproblems. These are then solved one by one and solutions are collated together. Example: MergeSort - algorithm



Closed form

- Generally speaking, recursive definition is more often a curse than a blessing; if possible, we'd want our functions to be defined directly via variable n so that the calculation would be quicker
    - This kind of formula is said to be in closed form
- Recursive definitions can often be decoded to closed form definitions
- This is done by calculating values using the recursive formula and then formulating a closed form formula using heuristics
- The closed form formula is then established correct - usually by using proof by induction

Example 3: Post-apocalypse

- Example inspired by sci-fi literature:
- Humans have, thanks to modern medicine, become immortal, but the fertility has collapsed due to pollution. As a solution, humankind has outsourced reproduction tasks for factories, which automatically produce human children in artificial tanks.
- Now a mystical virus has changed the majority of population to slow zombies, driving remaining humans to a handful of fortifications. Inside the reinforced concrete walls people can live in peace.
- Let's examine a fort, which is located near a certain baby factory. The leaders of the fort want to have a new generation of inhabitants every 15 years. For this reason, the current inhabitants have to visit the baby factory in order to "loot babies". Everyone (except the leaders) must participate, and each participant of the looting trip can grab two babies with them (one in each hand). Unfortunately when returning to the fort, one participant must remain outside in order to close the gate.
- In addition to the leaders, how many vassals (= not leaders) are there in the fort after *n* looting trips, if the initial number of vassals was a) 1 b) x ?

Example 3: Post-apocalypse

- If all vassals leave for the looting trip and carry two babies each to the fort, the number of vassals can be defined recursively as

$$p_{n+1} = 3p_n - 1$$

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$$p_1 = 3p_0 - 1 = 3 \cdot 1 - 1 = 2$$

$$p_2 = 3p_1 - 1 = 3 \cdot 2 - 1 = 5$$

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$$p_3 = 3p_2 - 1 = 3 \cdot 5 - 1 = 14$$

$$p_4 = 3p_3 - 1 = 3 \cdot 14 - 1 = 41$$

$$p_5 = 3p_4 - 1 = 3 \cdot 41 - 1 = 122$$

- Doesn't look good – no simple heuristic is found yet.

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$$p_1 = 3x - 1$$

$$p_2 = 3(3x - 1\ ) - 1 = 9x - 3 - 1$$

$$p_3 = 3(9x - 3 - 1) - 1 = 27x - 9 - 3 - 1$$

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- Now we can see some logic in here! The closed form could look like this:

$$p_n = 3^n x - 3^{n-1} - 3^{n-2} - \dots - 3^0$$

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- Let's now simplify this a bit further:

$$p_n = 3^n x - \underbrace{3^{n-1} - 3^{n-2} - \cdots - 3^0}_{\text{Write using sum notation}}$$

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Geometric sum,
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Sum formula:

$$S_n = \frac{1 - q^n}{1 - q}$$

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Geometric sum, where q = 3. Sum formula: $S_n = \frac{1 - q^n}{1 - q}$

$$p_n = 3^n x - \frac{1 - 3^n}{1 - 3} = 3^n x + \frac{1}{2} - \frac{3^n}{2} = 3^n \left(x - \frac{1}{2} \right) + \frac{1}{2}$$

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- The candidate for closed form formula of this recursively defined sequence is therefore

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- Is it? Well, let's prove the correctness of this formula using proof by induction
- Basic step: When $n = 0$, we should get $p_0 = x$

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BS:

$$p_0 = 3^0\left(x - \frac{1}{2}\right) + \frac{1}{2} = 1\left(x - \frac{1}{2}\right) + \frac{1}{2} = x - \frac{1}{2} + \frac{1}{2} = x$$

- Seems to work. Basic step is ok!

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- Then we make the induction hypothesis:

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$$= 3^{k+1}\left(x - \frac{1}{2}\right) + \frac{3}{2} - 1 = 3^{k+1}\left(x - \frac{1}{2}\right) + \frac{1}{2}$$

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IH: $$p_k = 3^k\left(x - \frac{1}{2}\right) + \frac{1}{2}$$

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IS: $$p_{k+1} = \boxed{3^{k+1}\left(x - \frac{1}{2}\right) + \frac{1}{2}}$$

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$$p_{k+1} = 3p_k - 1 = 3\left(3^k\left(x - \frac{1}{2}\right) + \frac{1}{2}\right) - 1$$

$$= 3^{k+1}\left(x - \frac{1}{2}\right) + \frac{3}{2} - 1 = \boxed{3^{k+1}\left(x - \frac{1}{2}\right) + \frac{1}{2}}$$

Same, so the IS claim is correct!

- Conclusion: The closed form formula is correct!

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- Therefore, as a solution for the problem we can write:

- b) If initially there are x vassals in the fort, then after *n* looting trips the number of vassals is

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$$p_n = 3^n\left(1 - \frac{1}{2}\right) + \frac{1}{2} = \frac{3^n}{2} + \frac{1}{2} = \frac{3^n + 1}{2}$$

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- Let's calculate a couple of values just to make sure:

$$p_1 = \frac{3^1 + 1}{2} = \frac{4}{2} = 2 \qquad p_2 = \frac{3^2 + 1}{2} = \frac{10}{2} = 5$$

$$p_3 = \frac{3^3 + 1}{2} = \frac{28}{2} = 14 \qquad p_4 = \frac{3^4 + 1}{2} = \frac{82}{2} = 41$$

Thank you!