Relations, part 2

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Properties of relations

- A relation defined in set X may have certain properties which have their own special terms
- Relations can be
  - Reflexive (or irreflexive)
  - Symmetric (or antisymmetric)
  - Transitive
  - Comparable
- NOTE! A relation doesn’t need to be any of these
  - The larger the number of domain elements, the more common it is that none of these properties exist
- Let’s examine these terms a bit more in detail – what they mean and how can we recognize them

Notes on terminology

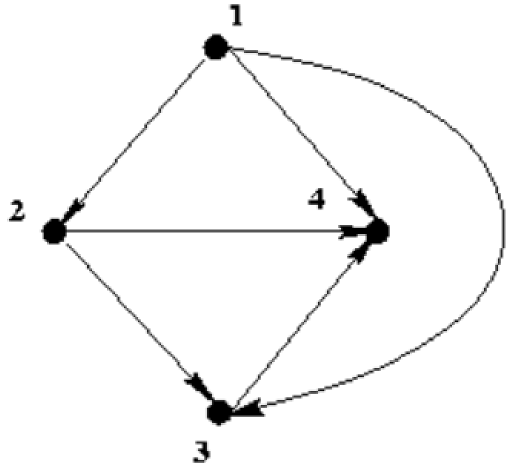
- Elements of a (2-place) relation are ordered pairs

$$R = \{(1,4), (3,2), (3,5)\}$$ (R here has 3 elements.)

- Often when we're talking about elements in relations, we mean the elements in its domain and range
  - These are single values
- In order to clarify (often surprisingly unambiguous) terminology, we try to specify when we're talking about domain elements or range elements
- In many cases, our domain = range, which causes more confusion, because same elements are both domain and range elements
  - …let's just say "domain elements" in general

Notes on terminology

- Digraphs have *nodes* and *arrows*
  - Nodes in a digraph = domain elements
  - Arrows in a digraph = relation elements (ordered pairs)
- When we're talking about relation matrices, term "element" refers to matrix elements
  - …which is kind of accurate, because each matrix element that's 1 corresponds to a relation element (ordered pair)
- So, digraph arrows = relation matrix 1-elements



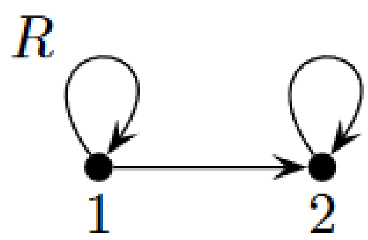
$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6 arrows in our digraph
→ 6 pcs of 1-elements in the relation matrix!

Reflexive relation

- A relation is said to be *reflexive*, if its every domain element is in relation to itself
  - Mathematically: $x\ R\ x \text{ for all } x \in X$
- This property is easy to spot
  - Digraph: every node has a loop arrow to itself
  - Relation matrix: diagonal consists of only 1s

$$M_R := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Irreflexive relation

- Respectively, a relation is *irreflexive* if none of its domain elements are in relation to themselves
  - Mathematically: $x \not{R} x \ for \ all \ x \in X$
- Also easy property to recognize:
  - Digraph: not a single node has a loop arrow to itself
  - Relation matrix: diagonal consists of only 0s

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- If some domain elements (but not all) are in relation to themselves, the relation is neither reflexive nor irreflexive

Symmetric relation

- A relation is said to be *symmetric*, if it holds in both directions
  - Mathematically: $x\ R\ y \ \Rightarrow y\ R\ x\ for\ all\ x, y \in X$
- Identification takes a bit longer look, but is not difficult
  - Digraph: if there is an arrow from node i to node j, then there has to be also an arrow from j to i*
  - Relation matrix: matrix is symmetric (about its diagonal)

$$M_S := \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

*Note: some authors use double arrows (one line, two arrowheads) in these cases.

Antisymmetric relation

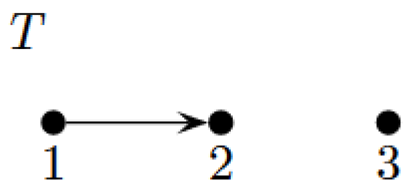
- Respectively, a relation is said to be *antisymmetric* if none of its domain elements are in two-way relation with each other
  - Mathematically: $x\ R\ y\ \wedge y\ R\ x \Rightarrow x = y\ \ for\ all\ x, y \in X$
- Identification as easy as in previous case
  - Digraph: if there is an arrow from node i to node j, then there is no arrow back from j to i
  - Relation matrix: if $a_{ij} = 1$, then $a_{ji} = 0$ (and vice versa)

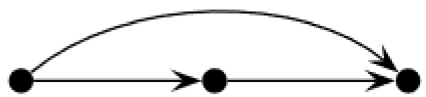
$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- If some nodes (but not all) are in relation to each other in both ways, the relation is not symmetric nor antisymmetric

Transitive relation

- A relation is said to be *transitive*, if all domain elements which are indirectly in relation to each other, are also directly in relation to each other
  - Mathematically: $x\ R\ y\ \wedge y\ R\ z \Rightarrow x\ R\ z\ \ for\ all\ x, y, z \in X$
- Harder to identify directly – needs some work
  - Digraph: if we can get from one node to another node using a route of n arrows, we can get there also directly via a single arrow
  - Relation matrix: examine the n x n-matrix raised to powers 2, 3, … , n-1 and focus on elements which are nonzero; if for all powers each element which is nonzero is also nonzero in the original matrix, relation is transitive





Transitive relation

- Matrix example: $$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Transitive relation

- Matrix example:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

  - 3x3-matrix
  - So, it’s enough to examine the 2nd power:

Transitive relation

- Matrix example:
  - 3x3-matrix
  - So, it's enough to examine the 2nd power:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_R^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Transitive relation

- Matrix example:
  $$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
  - 3x3-matrix
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$$M_R^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

  - Examination reveals that every nonzero element is also nonzero in the original matrix

Transitive relation

- Matrix example:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

  - 3x3-matrix
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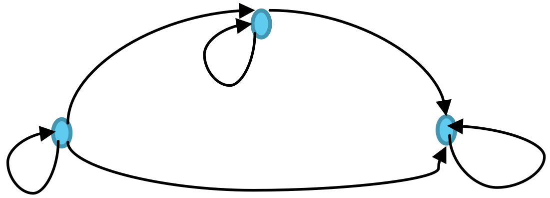
  - Examination reveals that every nonzero element is also nonzero in the original matrix
  - So, the relation is transitive
  - If the matrix would have been 4x4, then we had been forced to examine also M_R^3

Comparable relation

- A relation is said to be *comparable* if every pair of domain elements are comparable to each other
  - Mathematically: $x \preccurlyeq y \vee y \preccurlyeq x \ for\, all\, x, y \in X$
  - Plain English: we can choose any pair of (x,y) and determine the order of x and y
- Easy to understand, harder to spot
  - Digraph: all nodes are connected by (at least one) arrow and all nodes are in relation to themselves
  - Relation matrix: the union of relation matrix and its transpose consists of only 1s



- This kind of relation is also called strongly connected

Definitions for relation properties

- The previously mentioned properties are formally defined in such a way that relation R in set X is
  1) Reflexive IFF $I \subseteq R$
  2) Irreflexive IFF $R \cap I = \emptyset$
  3) Symmetric IFF $R^{-1} = R$
  4) Antisymmetric IFF $R \cap R^{-1} \subseteq I$
  5) Transitive IFF $R \circ R \subseteq R$
  6) Comparable IFF $R \cup R^{-1} = X \times X$

- Many properties can be proven in other ways, too (like from digraphs or relation matrices, as mentioned on previous slides), but those ways have been derived from these definitions

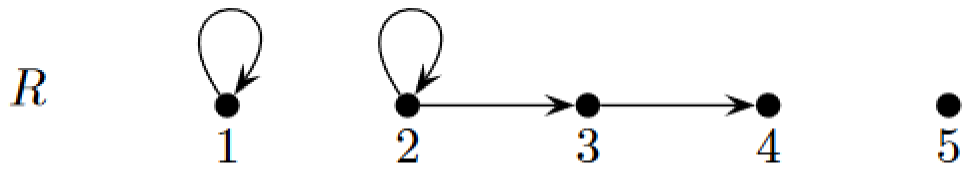
Closures

- Sometimes we want our relation to have some of the beforementioned properties
- If the relation does not possess this desired property, we can in some cases complete the original relation R to a new relation R' which has this property
- This completion is done by inserting additional elements (= ordered pairs) to the relation
  - In digraph: add arrows
  - In relation matrix: add 1s
- A relation can always be completed to be
  - Reflexive
  - Symmetric
  - Transitive
  - Comparable

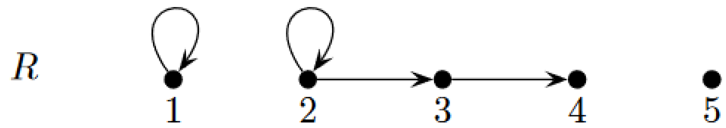
Closures

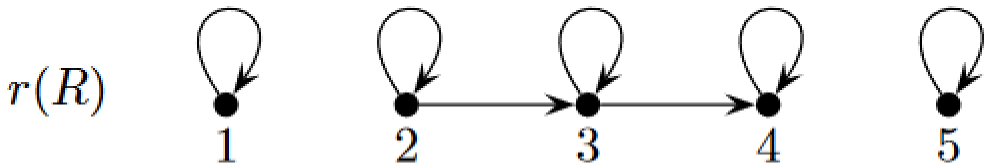
- The smallest possible completed relation R’ is called the *closure* of relation R
  - Usually it’s specifically mentioned about which property the completion is made – for example “reflexive closure”
  - Alternatively these can be marked by initial letters:
    - reflexive closure r(R)
    - symmetric closure s(R)
    - transitive closure t(R)
- Example: is the relation below reflexive, symmetric or transitive? If not, formulate the respective closure.

$$X = \{1,2,3,4,5\} \quad R = \{(1,1),(2,2),(2,3),(3,4)\}$$



Closures

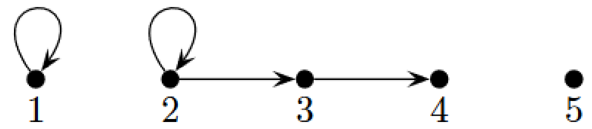




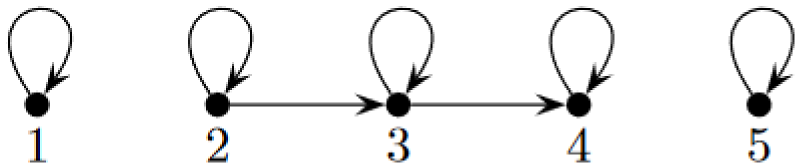
$$r(R) = \{(1,1), (2,2), (2,3), (3,3), (3,4), (4,4), (5,5)\}$$

Closures

R

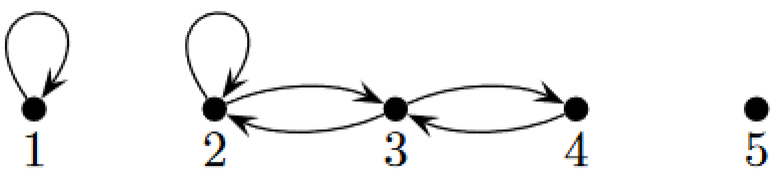


$r(R)$



$$r(R) = \{(1,1),(2,2),(2,3),(3,3),(3,4),(4,4),(5,5)\}$$

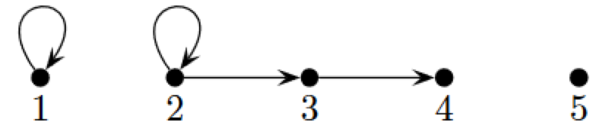
$s(R)$



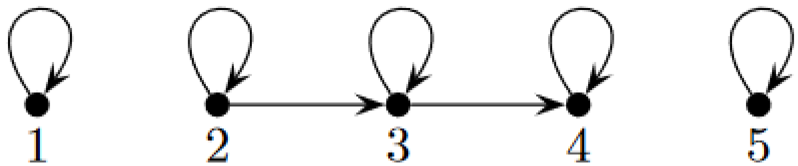
$$s(R) = \{(1,1),(2,2),(2,3),(3,2),(3,4),(4,3)\}$$

Closures

R

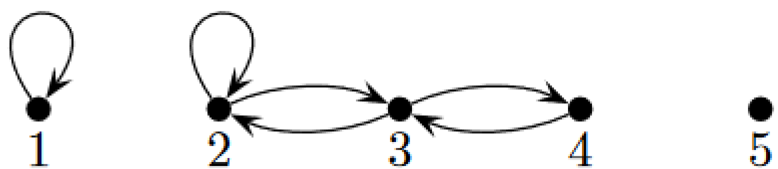


$r(R)$



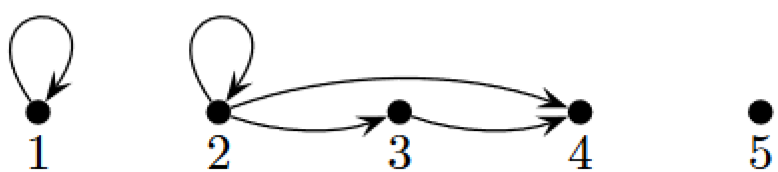
$$r(R) = \{(1,1),(2,2),(2,3),(3,3),(3,4),(4,4),(5,5)\}$$

$s(R)$



$$s(R) = \{(1,1),(2,2),(2,3),(3,2),(3,4),(4,3)\}$$

$t(R)$



$$t(R) = \{(1,1),(2,2),(2,3),(2,4),(3,4)\}$$

Equivalence relation

- We're often interested in classifying domain elements based on their properties
- For this reason, we need a construct called *equivalence relation*
- A relation defined in set X is an equivalence relation, if it possesses all these properties:
  - Reflexive
  - Symmetric
  - Transitive
- Equivalence relation is commonly marked by a tilde symbol (~)

Equivalence classes

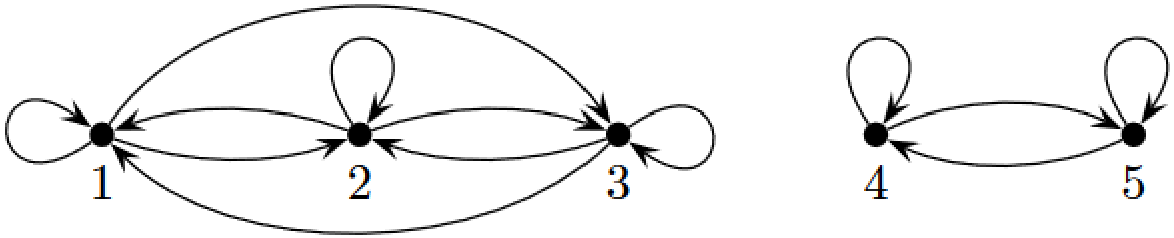
- All domain elements of an equivalence relation are not necessarily equivalent to all others, but they can form sets which are equivalent on their own
- This kind of sets are called *equivalence classes*
- For example, the equivalence class of domain element a is defined as

$$a/\sim = \{ x \in X \mid x \sim a \}$$

  - This means the set of all those domain elements that are equivalent to a
  - A domain element is always in equivalence at least to itself
- The number of equivalence classes is often interesting

Equivalence classes

- Let's examine the following relation, which is an equivalence:



- The equivalence class defined by node 1 is

$$1/\sim \; = \{\, x \in X \mid x \sim 1 \,\} = \{1, 2, 3\}.$$

- The equivalence class defined by node 2 is $2/\sim \; = \{1, 2, 3\}$
- The equivalence class defined by node 3 is $3/\sim \; = \{1, 2, 3\}$
- The equivalence class defined by node 4 is $4/\sim \; = \{4, 5\}$
- The equivalence class defined by node 5 is $5/\sim \; = \{4, 5\}$

Equivalence classes

- We notice the following things:
  - There are two equivalence classes
  - Equivalence classes are formed by domain elements which are equivalent between each other in such a way that each domain element belongs to exactly one equivalence class
  - In a digraph, equivalence classes are separated to different “islands” where arrows travel from every node to every other node of the same equivalence class
- Example: X is a set of students in school
  - Equivalence relation: $x \sim y \Leftrightarrow x \text{ and } y \text{ are on same class}$

    ⟶ number of equivalence classes = number of classes in school
  - Equivalence relation: $x \sim y \Leftrightarrow x \text{ and } y \text{ have same handedness}$

    ⟶ number of equivalence classes = 2 (left or right-handed)

Equivalence classes: Example

- Examine the following relation R in set $X = \{1,2,3,4,5,6\}$:

$$R = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5), (2,2), (2,6), (6,2), (6,6), (4,4)\}.$$

- a) Is R an equivalence relation?
- b) Draw a digraph. If R is an equivalence relation, how many equivalence classes does it have?
- c) If yes, define what these equivalence classes are.

Equivalence classes: Example

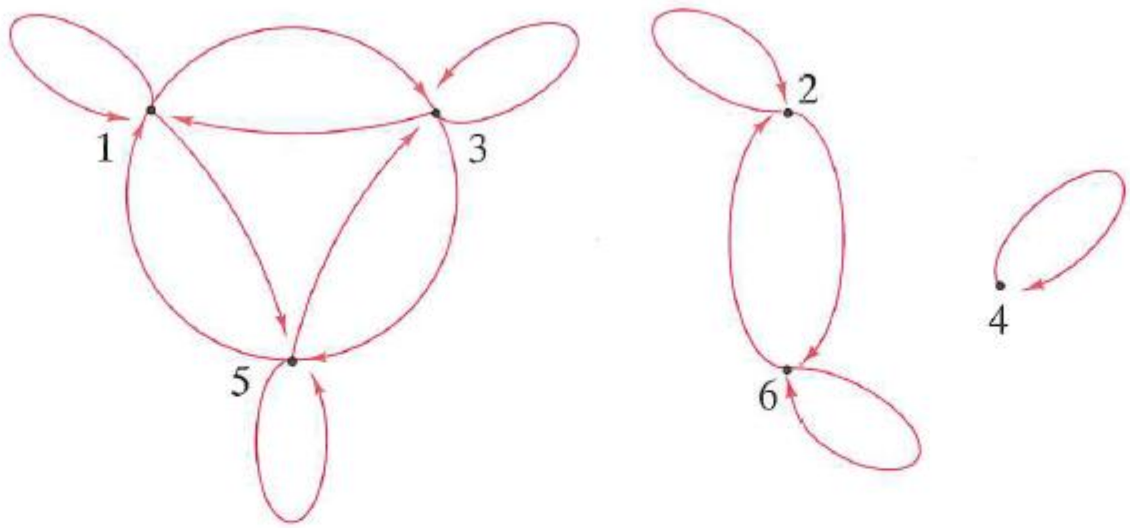
- Define the relation matrix:

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- This relation seems to be:
  - Reflexive (only 1s on the diagonal)
  - Symmetric (can be mirrored subject to its diagonal)
- In order to define whether R is transitive or not, we should compute 2nd, 3rd, 4th and 5th powers of M_R
  - Doesn't sound nice without Matlab; let's draw the digraph

Equivalence classes: Example

- Digraph:



- Seems to be transitive, so R is an equivalence relation!
- Number of equivalence classes seems to be three
- These classes are $\{1,3,5\}, \{2,6\}$ *and* $\{4\}$

Total order

- Another task that is often interesting is ordering the domain elements
- For this matter we define a new construct – an ordering relation called *total order*
- A relation defined in set X is a total order, if it possesses all these properties:
  - Reflexive
  - Antisymmetric
  - Transitive
  - Comparable
- The comparability is noted by the notation $x \preccurlyeq y$
  - “x precedes or is equal to y”

Total order

- These requisites have logical justifications:
  - Reflexive: each domain element precedes itself
  - Antisymmetric: two different domain elements cannot both precede each other
  - Transitive: precedence is an inherited property – if domain element a precedes b, then a precedes also all domain elements which come after b
  - Comparable: of two domain elements, one always precedes the other
- If the comparability requisite is not fulfilled, but other three are, then the relation is said to be a *partial order*
  - In a partial order, there are some domain elements which can’t be compared to each other (and hence, precedence can’t be deciphered)

Total order

- Set X, where we have total order, is an ordered set
- For example, the simplest way to order the set of integers is to set the relation as

$$x \preceq y \Leftrightarrow x \leq y$$

- Orders can be illustrated by using a *Hasse diagram*
  - Nodes which are in relation to each other are connected by a line in such a way that the earlier precedes the latter
  - Can be drawn from bottom to top or from left to right
  - Total order can be seen right away: all nodes are in line

Hasse diagram

- Example: examine the set $X = \{a, b, c\}$ and there the following relations R, S, T and U

$$R = \{(a,a),(b,b),(c,c)\}$$

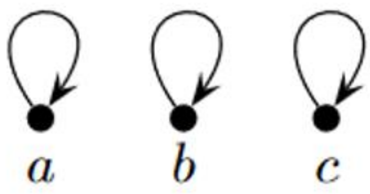
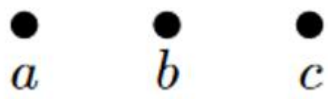
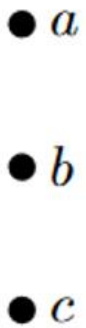
$$S = \{(a,a),(b,b),(c,c),(a,b)\}$$

$$T = \{(a,a),(b,b),(c,c),(a,b),(a,c)\}$$

$$U = \{(a,a),(b,b),(c,c),(a,b),(a,c),(b,c)\}$$

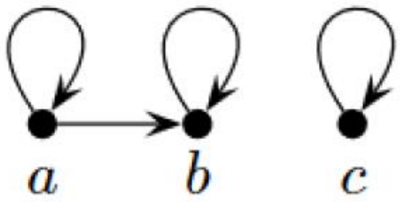
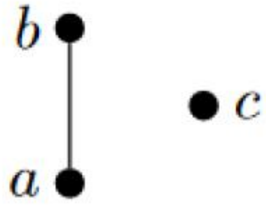
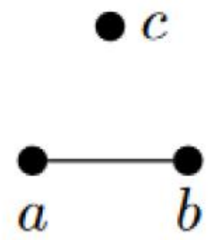
- Which of these are partial orders and which are total orders – or are they either of these?
- Draw digraphs and Hasse diagrams of both in order to find out

Hasse diagram

Relation	Digraph	Hasse diagram (bottom to top)	Hasse diagram (left to right)
R			

- We notice that R is reflexive, antisymmetric and transitive
- However, R is not comparable, so R is a partial order

Hasse diagram

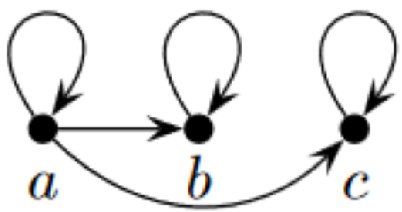
Relation	Digraph	Hasse diagram (bottom to top)	Hasse diagram (left to right)
S			

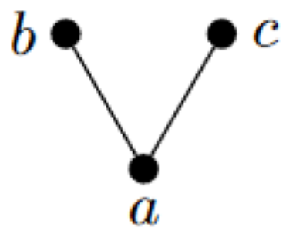
- We notice that the situation is the same for S
- So, S is a partial order

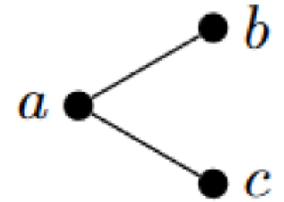
Hasse diagram

Relation | Digraph | Hasse diagram (bottom to top) | Hasse diagram (left to right)

T

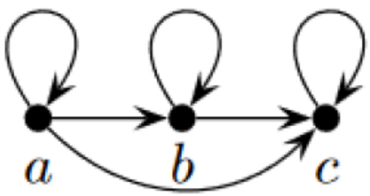
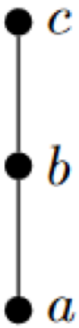
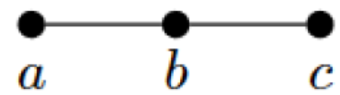






- Same goes for T: still not comparable (order of b and c is not clear, so they can't be compared)
- Hence, T is a partial order

Hasse diagram

Relation	Digraph	Hasse diagram (bottom to top)	Hasse diagram (left to right)
U			

- Relation U finally is comparable (on top of 3 other conditions)
- Therefore, U is a total order!

Thank you!