# Relations

Olli-Pekka Hämäläinen

# Ordered pair

- Earlier when we examined sets we only considered their contents (i.e. of which elements they consisted of), but disregarded the order of elements
- Let's now define a term *ordered pair*, where the elements form a two-element queue
- Notation: (x,y)
- Two ordered pairs are equal if their 1st elements are the same and 2nd elements are the same
  - Mathematically speaking:

$$(x, y) = (u, v) \Leftrightarrow x = u \wedge y = v$$

# Cartesian product

- The *Cartesian product* of sets A and B is defined as

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

- So, it is a set which consists of all ordered pairs (x,y) where x is a member of A and y is a member of B
- The number of elements (= ordered pairs) in this Cartesian product set we have already examined during the previous lecture
- For example, if $A = \{1,2,3\}$ and $B = \{4,5\}$, then

$$A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$$

$$B \times A = \{(4,1), (4,2), (4,3), (5,1), (5,2), (5,3)\}$$

- Note: Cartesian product is not commutative, because

  $(A \times B) \neq (B \times A)$

# Cartesian product

- The definition of a Cartesian product can be generalized to a case of multiple sets
- Hence, if there are n sets, the Cartesian product of these sets is a set which consists of n-element ordered queues - *n-tuples*

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, \dots, x_n) \quad | x_1 \in A_1 \wedge \cdots \wedge x_n \in A_n\}$$

- The number of elements (= n-tuples) in this set can be calculated using the multiplication principle
- For example, set $\mathbb{R}^2$ is the set of ordered pairs of real numbers – graphically represented as a plane
- Likewise, set $\mathbb{R}^3$ is the set of ordered triplets (3-tuples) of real numbers – graphically represented as 3-dimensional space

# Cartesian product rules

- So, Cartesian product is not commutative
- Associativity law is not applicable either, because now when the order matters, the elements of our set may contain ordered pairs inside ordered pairs

$$A \times (B \times C) = \{\, (x, (y, z)) \mid x \in A \land y \in B \land z \in C \,\}$$
$$(A \times B) \times C = \{\, ((x, y), z) \mid x \in A \land y \in B \land z \in C \,\}$$
$$A \times B \times C = \{\, (x, y, z) \mid x \in A \land y \in B \land z \in C \,\}$$

- Distributivity law on the other hand works for union, intersection and difference:

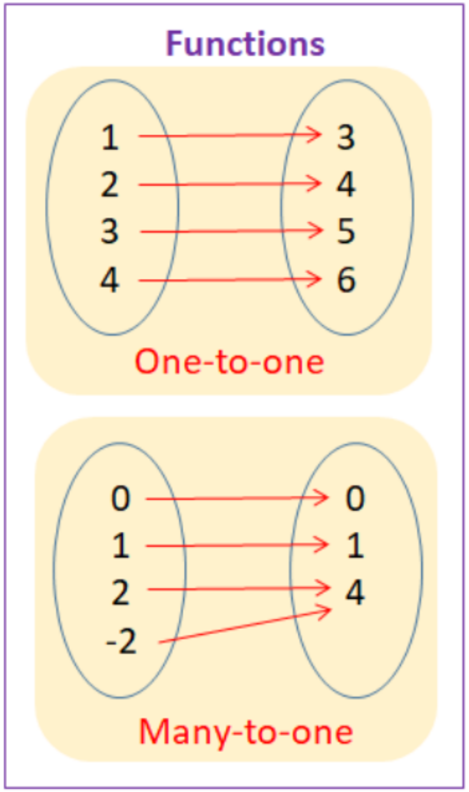
$$(1)\quad (A_1 \cup A_2) \times B = (A_1 \times B) \cup (A_2 \times B),$$
$$(2)\quad (A_1 \cap A_2) \times B = (A_1 \times B) \cap (A_2 \times B),$$
$$(3)\quad (A_1 \setminus A_2) \times B = (A_1 \times B) \setminus (A_2 \times B).$$

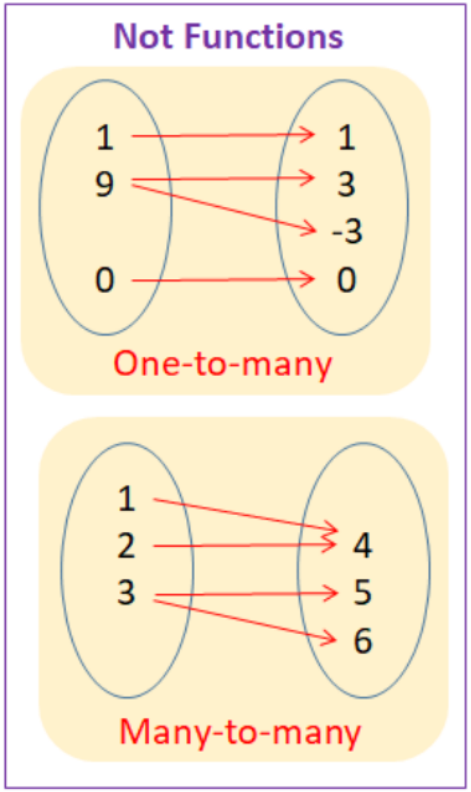
# Definition of a relation

- If a relation R is included in the Cartesian product $X \times Y$, we can say that R is a relation between X and Y
- Elements of the ordered pair ($x \in X$ and $y \in Y$) are in relation R to each other
- So, a two-place relation links together an element x from the domain and an element y from the range according to some rule
  - Notation: R(x,y) or $x\ R\ y$
- Relation can also be multi-place relation – for example R(x,y,z); in this case this domain-range-thinking is not applicable
- Two-place relations are most common, though

# Relation vs. function

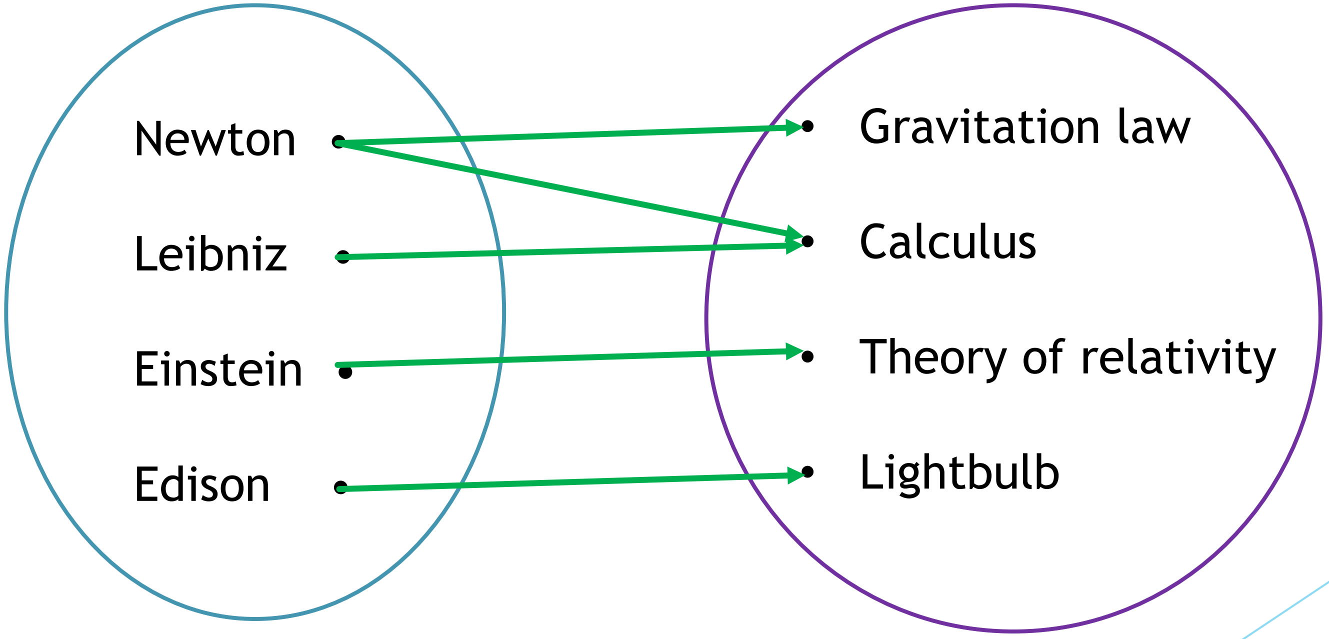
- A function y = f(x) is a special case of relation, where the function assigns each value x to exactly one value y
  - All cases below are relations, but only the ones on the left can be presented as functions



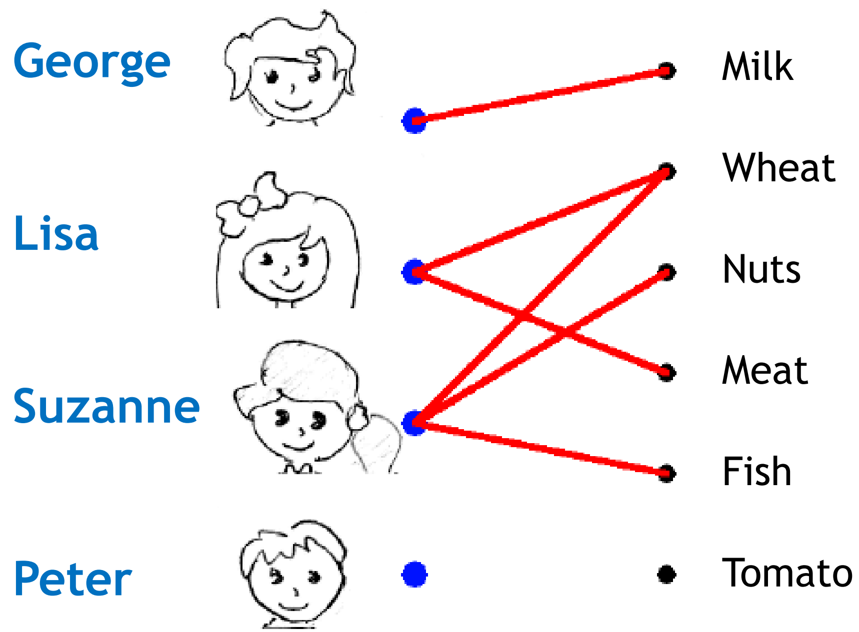
# Relation as a domain-range graph

- A two-place relation can be illustrated using a graph where we mark the domain elements and connect them to corresponding range elements by a relation arrow
- For example, relation $x\ R\ y \Leftrightarrow x\ has\ invented\ y$



# Relation as a domain-range graph

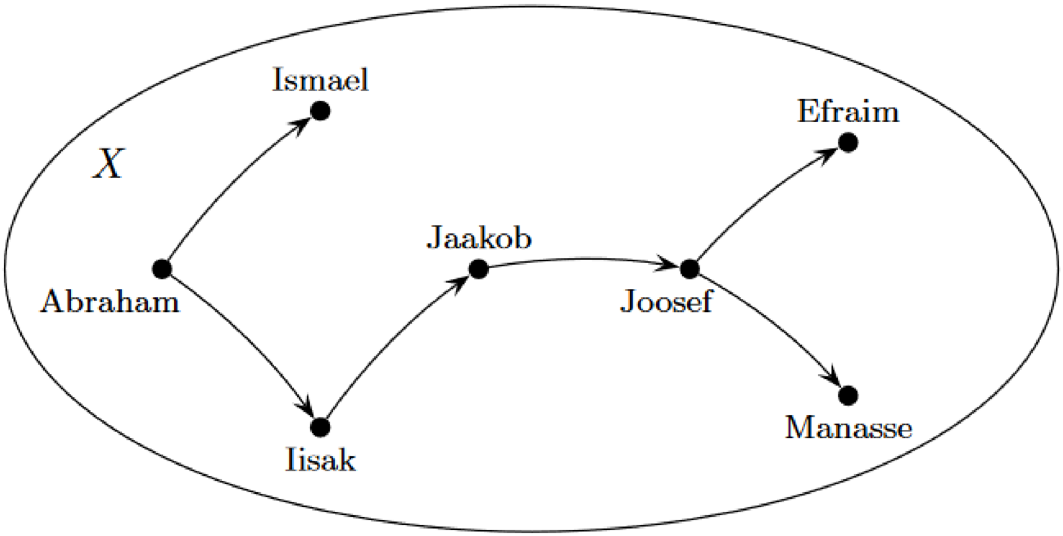
- This kind of relations can be seen often in applications whose background is not very mathematical




| | |
| --- | --- |
| Set X: children of a certain daycare group | Set Y: list of allergizing/avoided ingredients |

# Directed graph (digraph)

- If the domain and range have common elements, the smartest way of graphical representation is a *directed graph* – in short, a *digraph*
  - Gives a better picture of the relation
- Example relation from the bible: $x\ R\ y = x\ is\ y's\ father$




(Note: Names in Finnish spelling.)

# Directed graph (digraph)

- Another example: if in set X = {1,2,3,4} we define a relation $(x, y) \in R\ if\ x \leq y\ when\ x, y \in X$

# Directed graph (digraph)

- Another example: if in set X = {1,2,3,4} we define a relation $(x, y) \in R \text{ } if \text{ } x \leq y \text{ } when \text{ } x, y \in X$
- In this case, the ordered pairs of the relation are

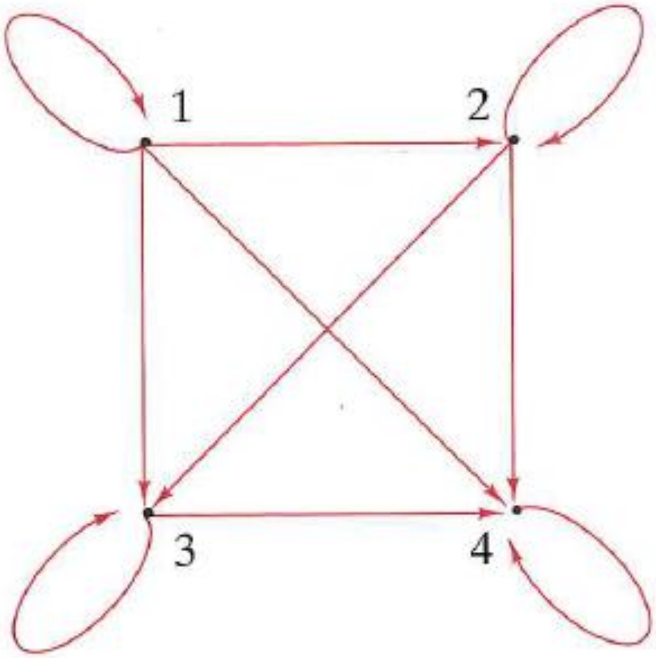
$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

# Directed graph (digraph)

- Another example: if in set X = {1,2,3,4} we define a relation $(x, y) \in R\ if\ x \leq y\ when\ x, y \in X$
- In this case, the ordered pairs of the relation are

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

- Digraph:



# Converse relation

- If we have a relation R from set X to set Y, it is natural that we can define the relation also contrariwise – so, from set Y to set X
- This kind of a relation in "opposite direction" is (logically) called a *converse relation* $R^{-1}$

$$y\,R^{-1}\,x \Leftrightarrow x\,R\,y$$

- For example, the previous inventor relation

$$x\,R\,y \;\Leftrightarrow x \text{ has invented } y$$

- ...has converse relation

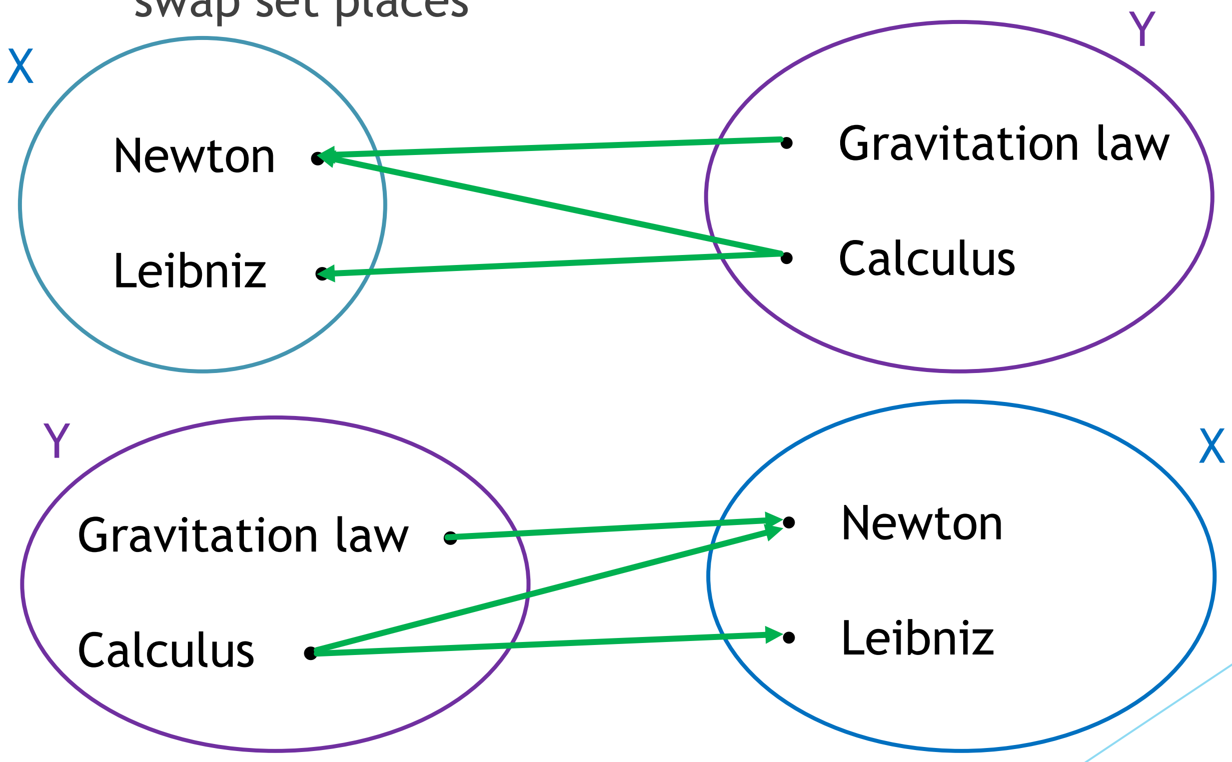
$$y\,R^{-1}\,x \;\Leftrightarrow y \text{ was invented by } x$$

- Respectively, the converse relation of the previous biblical fatherhood relation is

$$y\,R^{-1}\,x \;\Leftrightarrow y \text{ is a son of } x$$

# Converse relation

- The domain-range graph of the converse relation is logically the  same as the original – just the direction of arrows is inverted (same goes for digraphs)
- If we want the domain set on the left side, we have to swap set places

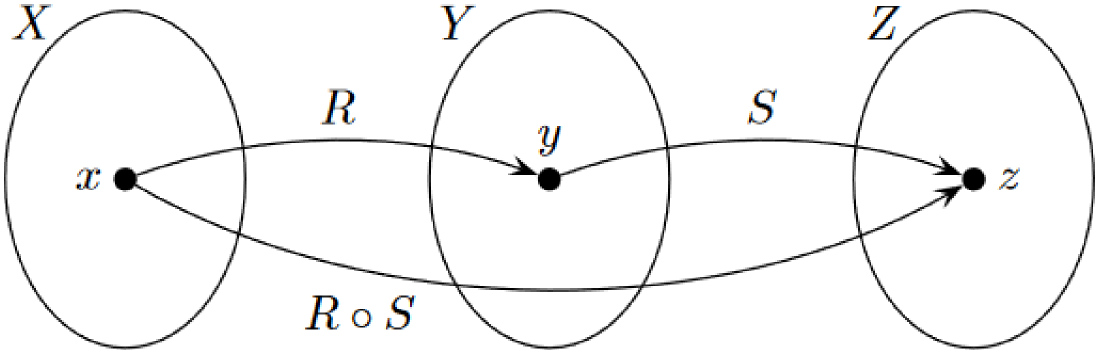


# Composition of relations

- Let's (vaguely) define relations:
  - R is a relation from set X to set Y
  - S is a relation from set Y to set Z
- Using these we can define a *composition*

$$x\ (R \circ S)\ z \Leftrightarrow \exists y \in Y\colon\ x\ R\ y \land y\ S\ z$$

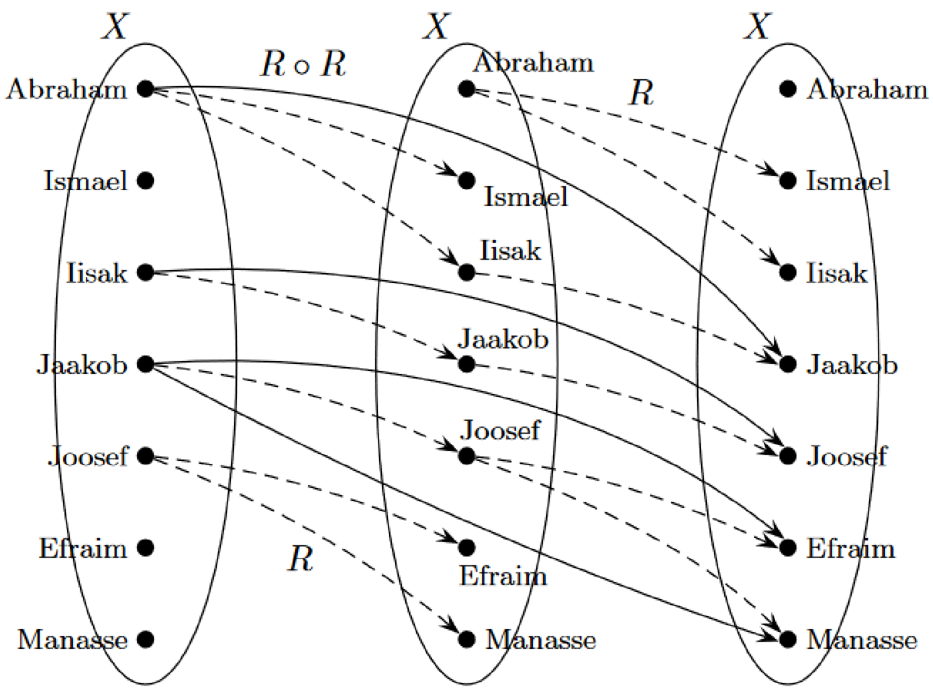
- So, elements of X and Z are in relation $R \circ S$ if and only if we can get from x to z via arrows in domain-range graph:



# Composition of relations

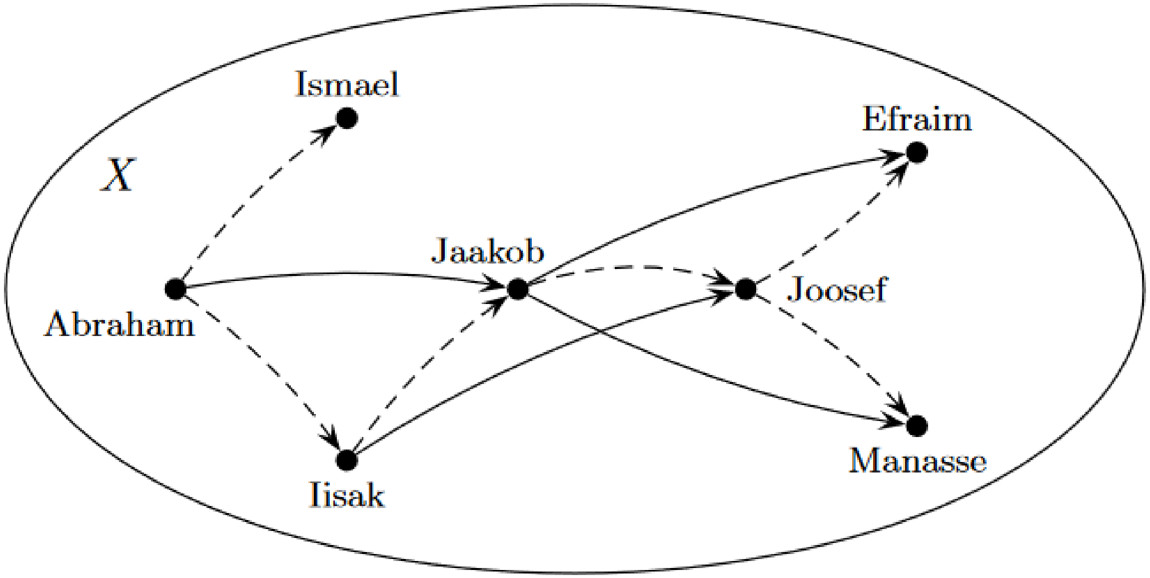
- We can also form a composition with the relation itself, so $R \circ R = R^2$
- In the case of previous biblical example, this relation would mean that

$$x\,R^2\,y \Leftrightarrow x\ is\ a\ grandfather\ of\ y$$



# Composition of relations

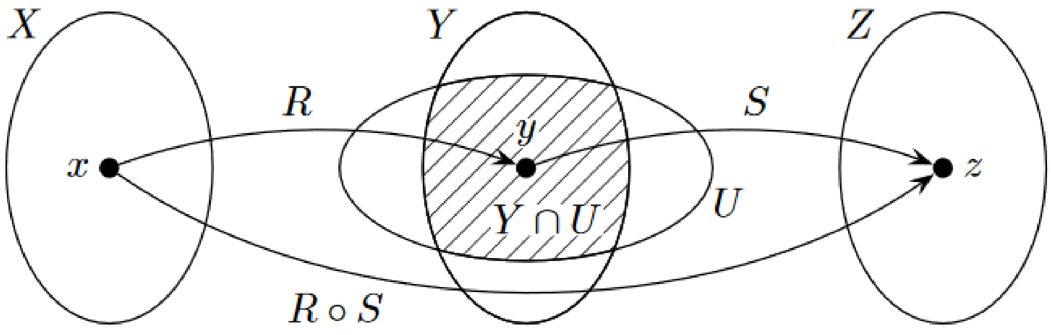
- In a digraph we can demonstrate this in such a way that an element x is in relation $R^2$ with element y if and only if we can get from x to y via a route of exactly two arrows



# Composition of relations

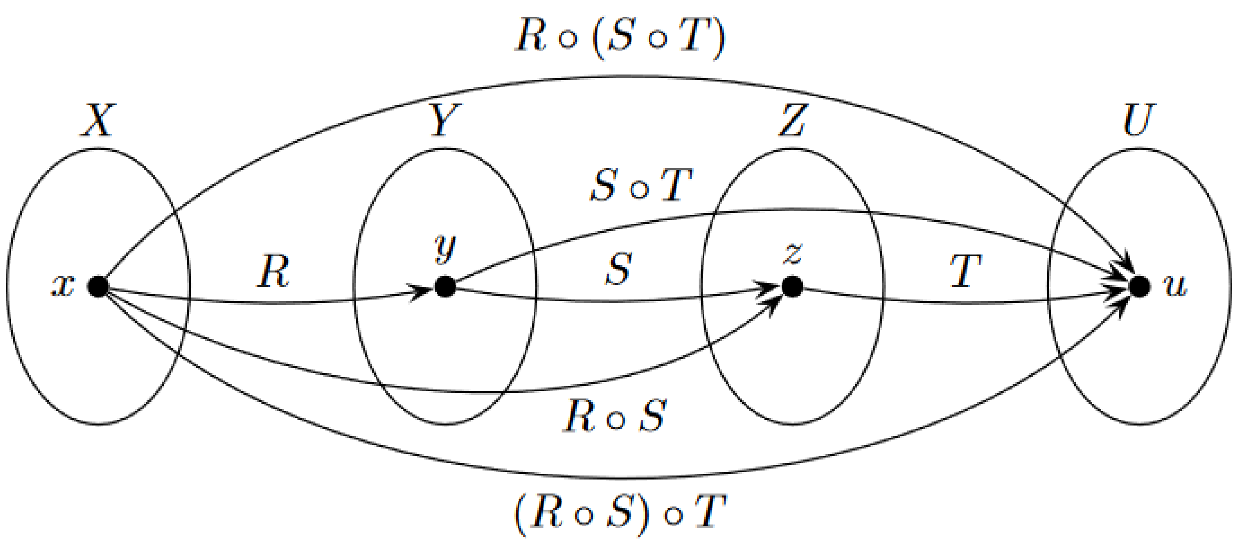
- At times, the requirement that the range of R and domain of S should be the same is a quite heavy constraint
- Luckily, this is not a hard constraint: we can freely combine relations as long as we set them a condition that the elements in domain of S (latter relation) must be included also in the range of the R (previous relation)

$$x(R \circ S)z \Leftrightarrow \exists y \in Y \cap U : xRy \wedge ySz$$



# Rules of relations

- Previous notions that we made for two-set relations can be generalized to calculation rules:
- Composition of relations are associative, so we can combine relations as we wish without thinking about parentheses (as long as the order remains the same!)



# Rules of relations

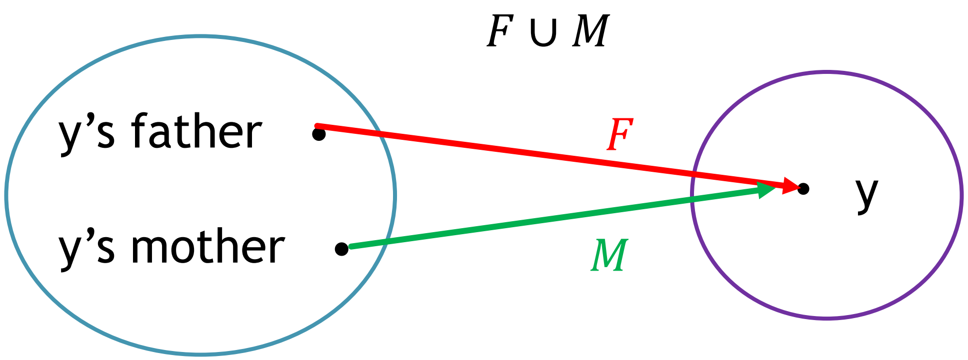
- Set theory operations (unions, intersections, etc.) can be performed to relations as well
- NOTE! “union of relations”(U) ≠ “composition of relations” (○)
- For example, if we have relations

$$x\ F\ y \Leftrightarrow x\ is\ y's\ father$$

$$x\ M\ y \Leftrightarrow x\ is\ y's\ mother$$

- Then the union of these relations is

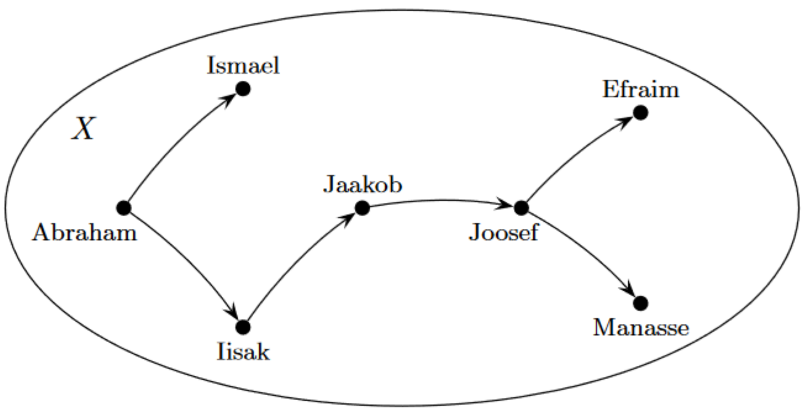
$$x\ (F \cup M)y \Leftrightarrow x\ F\ y \vee x\ M\ y \Leftrightarrow x\ is\ a\ parent\ of\ Y$$



# Rules of relations

- If R is a relation defined in set X, then its n-times composition is $R^n$
- We don't need to draw a separate digraph for these – we can use the digraph of the original relation:
  - $x\ R^n\ y$ if and only if we can get from element x to element y using a route which consists of n arrows
- For example, the ordered pairs which fulfill the previous biblical fatherhood relation $x\ R^3\ y \Leftrightarrow x\ is\ y's\ greatgrandfather$ are

$$\{(Abraham, Joosef), (Iisak, Efraim), (Iisak, Manasse)\}$$



# Relation matrix

- The internal relations between elements can be presented also using matrices
  - Two-place relation: domain elements are set as rows and range elements are set as columns
  - If there’s a relation between domain element i and range element j, then the element $M_{Rij}$ of a *relation matrix* gets a value 1 (if no relation, then 0)
- This is easiest when elements have number values, because then the row and column order is self-evident
  - If the elements are not number values, then the row and column order can be selected freely; this doesn’t make calculations any more complicated, but hinders the interpretation of results
- Relation matrix is the easiest way to depict relations to a computer

# Relation matrix

- Example: We have sets X = {1,2,3,4,5} and Y = {6,7,8,9,10}
- Define relation $x \ R \ y \Leftrightarrow x \ is \ a \ factor \ of \ y$

# Relation matrix

- Example: We have sets X = {1,2,3,4,5} and Y = {6,7,8,9,10}
- Define relation $x\ R\ y \Leftrightarrow x\ is\ a\ factor\ of\ y$
- The ordered pairs which specify the relation are

$$R = \{(1,6),\ (1,7),\ (1,8),\ (1,9),\ (1,10),\ (2,6),\ (2,8),\ (2,10),\ (3,6),\ (3,9),\ (4,8),\ (5,10)\}$$

# Relation matrix

- Example: We have sets X = {1,2,3,4,5} and Y = {6,7,8,9,10}
- Define relation $x \; R \; y \Leftrightarrow x \; is \; a \; factor \; of \; y$
- The ordered pairs which specify the relation are

$$R = \{(1,6),\ (1,7),\ (1,8),\ (1,9),\ (1,10),\ (2,6),\ (2,8),\ (2,10),\ (3,6),\ (3,9),\ (4,8),\ (5,10)\}$$

- Make a table:

| | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 |
| 3 | 1 | 0 | 0 | 1 | 0 |
| 4 | 0 | 0 | 1 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1 |

# Relation matrix

- Example: We have sets X = {1,2,3,4,5} and Y = {6,7,8,9,10}
- Define relation $x \; R \; y \Leftrightarrow x \; is \; a \; factor \; of \; y$
- The ordered pairs which specify the relation are

$$R = \{(1,6), (1,7), (1,8), (1,9), (1,10), (2,6), (2,8), (2,10), (3,6), (3,9), (4,8), (5,10)\}$$

- Make a table:

| | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 |
| 3 | 1 | 0 | 0 | 1 | 0 |
| 4 | 0 | 0 | 1 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1 |

Relation matrix:

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Calculations using relation matrices

- If we present the relations using relation matrices, the calculation of unions, intersections and compositions becomes surprisingly easy
- Relation matrices have only 1s and 0s as elements
- Remember Boolean algebra and define the following notations:
  - Boolean addition $\oplus$
  - Boolean element-wise multiplication $\otimes$
  - Boolean multiplication $\odot$
- Boolean addition is a familiar concept, but the two different multiplications may feel confusing
  - Let's check these out first!

# Boolean element-wise multiplication

- "Element-wise" multiplication means that the matrices are not multiplied as we've learned to do before in matrix calculation, but we just multiply the corresponding elements
  - In Matlab, this operator is the "dot-multiplication" .*
- "Boolean" prefix means just that 1x1 yields 1 and others (0x1, 1x0, 0x0) yield us a zero
  - Well, this is no different than in regular algebra
- So, the elements of resulting matrix $C = A \otimes B$ will be

$$c_{ij} = a_{ij} \otimes b_{ij}$$

- NOTE! This requires that the matrices A and B must be of same size (because otherwise some elements are left without a corresponding element)

# Boolean multiplication

- “Boolean multiplication” is performed the same way as regular matrix multiplication, but we follow Boolean addition law when summing up the results of row-by-column-multiplications
  - i.e. 1 +1 = 1
- Remembering this is sometimes hard for students who are not yet used to working in Boolean (after 10+ years, brains have gotten comfortable with regular algebra)
- If a student wishes to do these in an easier way, we can get the exact same results this way:
  - Calculate the matrix multiplication as before
  - Change all elements larger than 1 to 1s
  - Done!

# Calculations using relation matrices

- If we define relations R and S followingly:
  - R is a relation from set X to set Y
  - S is a relation from set Y to set Z
- The union, intersection and composition of these relations can be calculated via relation matrices in the following way:

$$M_{R\cup S} = M_R \oplus M_S$$

$$M_{R\cap S} = M_R \otimes M_S$$

$$M_{R\circ S} = M_R \odot M_S$$

- Converse relation matrix can be defined, too: it's just a transpose (NOTE! Not inverse!) of the original relation matrix:

$$M_{R^{-1}} = M_R^T$$

Note: Some authors use the notation $R^T$ for the converse relation in order to avoid confusion, but $R^{-1}$ is more common.

# Example 1

- Define matrices A, B and C:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

- Calculate $A \oplus B$, $A \otimes B$ and $A \odot C$.

# Example 1

- Define matrices A, B and C:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

- Calculate $A \oplus B$, $A \otimes B$ and $A \odot C$.

$$A \oplus B = \begin{pmatrix} 1 \oplus 0 & 1 \oplus 1 & 0 \oplus 0 \\ 0 \oplus 0 & 1 \oplus 0 & 0 \oplus 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

# Example 1

- Define matrices A, B and C:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

- Calculate $A \oplus B$, $A \otimes B$ and $A \odot C$.

$$A \oplus B = \begin{pmatrix} 1 \oplus 0 & 1 \oplus 1 & 0 \oplus 0 \\ 0 \oplus 0 & 1 \oplus 0 & 0 \oplus 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 1 \otimes 0 & 1 \otimes 1 & 0 \otimes 0 \\ 0 \otimes 0 & 1 \otimes 0 & 0 \otimes 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

# Example 1

- Define matrices A, B and C:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

- Calculate $A \oplus B$, $A \otimes B$ and $A \odot C$.

$$A \oplus B = \begin{pmatrix} 1 \oplus 0 & 1 \oplus 1 & 0 \oplus 0 \\ 0 \oplus 0 & 1 \oplus 0 & 0 \oplus 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 1 \otimes 0 & 1 \otimes 1 & 0 \otimes 0 \\ 0 \otimes 0 & 1 \otimes 0 & 0 \otimes 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A \odot C = \begin{pmatrix} (1 \otimes 1) \oplus (1 \otimes 0) \oplus (0 \otimes 0) & (1 \otimes 1) \oplus (1 \otimes 1) \oplus (0 \otimes 1) \\ (0 \otimes 1) \oplus (1 \otimes 0) \oplus (0 \otimes 0) & (0 \otimes 1) \oplus (1 \otimes 1) \oplus (0 \otimes 1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 \oplus 0 \oplus 0 & 1 \oplus 1 \oplus 0 \\ 0 \oplus 0 \oplus 0 & 0 \oplus 1 \oplus 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

# Example 2

- Define sets X = {1,2,3}, Y = {a,b,c} and Z = {q,r}.
  - a) Define matrices to relations x R y and y S z

$R = \{(1,a),(1,b),(2,b),(2,c),(3,a)\}$ $\quad S = \{(b,q),(c,q),(c,r)\}$

  - b) Define matrices for converse relation of R and composition $R \circ S$

# Example 2

- Define sets X = {1,2,3}, Y = {a,b,c} and Z = {q,r}.
  - a) Define matrices to relations x R y and y S z

$R = \{(1,a),(1,b),(2,b),(2,c),(3,a)\}$ $S = \{(b,q),(c,q),(c,r)\}$

  - b) Define matrices for converse relation of R and composition $R \circ S$

$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

# Example 2

- Define sets X = {1,2,3}, Y = {a,b,c} and Z = {q,r}.
  - a) Define matrices to relations x R y and y S z

$R = \{(1,a),(1,b),(2,b),(2,c),(3,a)\}$ $\quad S = \{(b,q),(c,q),(c,r)\}$

  - b) Define matrices for converse relation of R and composition $R \circ S$

$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$M_R M_S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 0 \end{pmatrix}$$

Change to “1”!

# Example 2

- Define sets X = {1,2,3}, Y = {a,b,c} and Z = {q,r}.
  - a) Define matrices to relations x R y and y S z

$R = \{(1, a), (1, b), (2, b), (2, c), (3, a)\}$ $S = \{(b, q), (c, q), (c, r)\}$

  - b) Define matrices for converse relation of R and composition $R \circ S$

$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$M_R M_S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 0 \end{pmatrix}$$

Change to "1"!

$$M_{S \circ R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

# Example 2

- Define sets X = {1,2,3}, Y = {a,b,c} and Z = {q,r}.
  - a) Define matrices to relations x R y and y S z

$R = \{(1,a),(1,b),(2,b),(2,c),(3,a)\}$ $\quad S = \{(b,q),(c,q),(c,r)\}$

  - b) Define matrices for converse relation of R and composition $R \circ S$

$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$M_R M_S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$M_{S \circ R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$M_{R^{-1}} = M_R^T = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

# Thank you!