1. We get the last digit of 7^{150} by finding its remainder when divided by 10:

$$7^{150} \equiv (7^2)^{75} \equiv 49^{75} \equiv (-1)^{75} = -1 \equiv 9 \text{ mod } 10.$$

This means that the last digit is 9.

2. We can select $x = 13$. Then $6x - 3 = 6 \cdot 13 - 3 = 78 - 3 = 75$. The following numbers are congruent with 75 modulo 17:

$$7 \equiv 24 \equiv 41 \equiv 58 \equiv 75 \equiv 92 \equiv \cdots$$

3. Let us denote the first three selected numbers by s_1, s_2, s_3. By the Division Theorem:

$$\begin{aligned} s_1 &= k_1 \cdot 3 + r_1, \\ s_2 &= k_2 \cdot 3 + r_2, \\ s_3 &= k_3 \cdot 3 + r_3, \end{aligned}$$

where $0 \leq r_1, r_2, r_3 < 3$. This gives that

$$s_1 + s_2 + s_3 = (k_1 + k_2 + k_3)3 + (r_1 + r_2 + r_3)$$

The only way that $s_1 + s_2 + s_3$ is divisible by 3 is when $r_1 + r_2 + r_3$ is divisible by 3. We have the following remainders:

$$\begin{aligned} 71 &\equiv 2 \text{ mod } 3 \\ 76 &\equiv 1 \text{ mod } 3 \\ 80 &\equiv 2 \text{ mod } 3 \\ 82 &\equiv 1 \text{ mod } 3 \\ 91 &\equiv 1 \text{ mod } 3 \end{aligned}$$

This means that we must select 76, 82, 91. Note that 76 + 82 is divisible by 2, so we can select the first 3 digits in this order.

The sum $76 + 82 + 91$ is odd, but after adding the fourth number, the sum must be even – because it must be divisible by 4. This means that next we must insert 71. Note that the sum

$$76 + 82 + 91 + 71 = 320$$

is divisible by 4. The **last** number to insert is 80.

4. We have that

$$6! = 2^4 \cdot 3^2 \cdot 5,$$

which means that $6! \equiv 0 \text{ mod } 9$. This implies that $k! \equiv 0 \text{ mod } 9$ for all $6 \leq k \leq 999$. Now

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \\ 3! &= 6 \\ 4! &= 24 \equiv 6 \text{ mod } 9 \\ 5! &\equiv 5 \cdot 6 = 30 \equiv 3 \text{ mod } 9 \end{aligned}$$

The remainder of

$$1! + 2! + 3! + 4! + 5! + 6! + \cdots + 999!$$

divided by 9 is 0, because

$$1 + 2 + 6 + 6 + 3 = 18 \equiv 0 \bmod 9.$$

5. Clock works "modulo 24" with respect to hours. The plane arrives to Peking at

$$18{:}10 + 8{:}30 = 26{:}40 \equiv 2{:}40 \bmod 24h$$

Stockholm time. Because Peking time is 7 hours ahead Stockholm time, the time in Peking is

$$2{:}40 + \text{ 7 hours } = 9{:}40.$$

6. Let a, b and $c > 0$ be integers such that $a \equiv b \bmod c$. This means that there are s, t, and $0 \leq r < n$ such that

$$a = sc + r \qquad \text{and} \qquad b = tc + r.$$

Then

$$a^2 = (sc + r)^2 = s^2c^2 + 2scr + r^2 = c(s^2c + 2sr) + r^2.$$

Similarly,

$$b^2 = (tc + r)^2 = t^2c^2 + 2tcr + r^2 = c(t^2c + 2tr) + r^2.$$

This means that $a^2 \equiv b^2 \bmod c$. By repeating this, we have that $a^n \equiv b^n \bmod c$ for all $n \geq 1$.

Because $2 \equiv 9 \bmod 7$, we have $2^n \equiv 9^n \bmod 7$. This implies that

$$2^n + 6 \cdot 9^n \equiv 9^n + 6 \cdot 9^n \equiv 7 \cdot 9^n \equiv 0 \bmod 7.$$