Functions and cardinality of sets

A function is a binary relation between two sets that associates to **each element** of the first set **exactly one** element of the second set. For example, the formula for the area of a circle, $A = \pi r^2$, gives the *area* A as a function of the *radius* r. The cardinality of a set is a measure of a set's size, meaning the number of elements in the set. Two sets A and B have a same number of elements only when elements of A can be paired in *one-to-one* correspondence with elements of B. This can be formulated by using functions called *bijections*. We will show that the number of decimal numbers has the same cardinality as the set of integers. But the size of real numbers is bigger than decimal numbers. We will shot that the interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is "as big" as the whole set of real numbers $\mathbb{R}$!

Definition 1. A binary relation $f \subseteq A \times B$ is said to be a **function** from a set A to set B if

(a) $(\forall a \in A)(\exists b \in B)\ afb$

(b) $(\forall a \in A)(\forall b_1, b_2 \in B)\ afb_1 \wedge afb_2 \implies b_1 = b_2$

Functions are also called **maps** or **mappings**.

In the definition of a function, A and B are called the **domain** and the **codomain** of the function f. If (x, y) belongs to the relation f, then y is the image of x under f. The f-value of x is denoted by $f(x)$. Note that in the above definition, (i) means that each element of A has an image, and (ii) says that this image is unique, meaning that a same x cannot have two images. In the following examples, we consider briefly some common real functions.

Example 2 (Exponents)**.** Exponentiation is written as b^n. The **base** b and the **exponent** or **power** n ("b raised to the power of n"). When n is a positive integer, exponentiation corresponds to repeated multiplication of the base:

$$b^n = \underbrace{b \times \cdots \times b}_{n \text{ times}}.$$

The exponent is usually shown as a superscript to the right of the base.

One has $b^1 = b$. To extend this property to non-positive integer exponents, b^0 is defined to be 1, and b^{-n} (with n a positive integer and b not zero) is defined as $\frac{1}{b^n}$. In particular, b^{-1} is equal to $1/b$.

For any positive integers m and n, one has

$$\begin{aligned} b^{m+n} &= b^m \cdot b^n \\ (b^m)^n &= b^{m \cdot n} \\ (b \cdot c)^n &= b^n \cdot c^n \end{aligned}$$

Example 3 (Roots as rational exponents)**.** If x is a nonnegative real number and n is a positive integer, $x^{\frac{1}{n}}$ of $\sqrt[n]{x}$ denotes the unique positive real nth root of x, that is, the unique positive real number y such that $y^n = x$.

If x is a positive real number and $\frac{p}{q}$ is a rational numbers, then $x^{\frac{p}{q}}$ is defined as

$$x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}} = (x^{\frac{1}{q}})^p.$$

The equality on the right may be derived by setting $y = x^{\frac{1}{q}}$ and writing

$$(x^{\frac{1}{q}})^p = y^p = ((y^p)^q)^{\frac{1}{q}} = ((y^q)^p)^{\frac{1}{q}} = (x^p)^{\frac{1}{q}}.$$

If r is a positive rational number, $0^r = 0$ by definition.

For instance, $5/8 = 0.625$. Therefore, $2^{0.625} = 2^{\frac{5}{8}} = (2^5)^{\frac{1}{8}} = 32^{\frac{1}{8}} \approx 1.54221$. We are dealing with approximations so $1.54221^8 \approx 31.99986$.

Example 4. If $x = \pi$, the non-terminating decimal representation $\pi = 3.14159\ldots$ and the monotonicity of the rational powers can be used to obtain intervals bounded by rational powers that are as small as desired, and must contain b^π. Here are examples of such intervals:

$$\left[b^3, b^4\right], \left[b^{3.1}, b^{3.2}\right], \left[b^{3.14}, b^{3.15}\right], \left[b^{3.141}, b^{3.142}\right], \left[b^{3.1415}, b^{3.1416}\right], \left[b^{3.14159}, b^{3.14160}\right], \ldots$$

We will study **limits** later in this course.

A **power function** is a real function that can be represented in the form

$$f(x) = a \cdot x^p$$

where a and p are fixed real numbers and x is the variable. The value a is called the **coefficient** and p is the **power**. The *power function* is closely related to *exponent function*: A power function contains a variable base raised to a fixed power, but the exponent function has a constant base raised to a variable power. For instance, $f(x) = x^2$ is a power function. A

Example 5 (Absolute values)**.** For any real number x, the **absolute value** of x is denoted by $|x|$. It is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

The absolute value of x is thus always ≥ 0. If x itself is negative ($x < 0$), then its absolute value is necessarily positive ($|x| = -x > 0$). The absolute value of x can be expressed also as

$$\sqrt{x^2}$$

What is *essential* is that the absolute value of the difference of two real numbers is the **distance** between them.

The absolute value has the following four fundamental properties (a and b are real numbers):

(a) $|a| \geq 0$

(b) $||a|| = |a|$

(c) $|-a| = |a|$

(d) $|ab| = |a|\,|b|$

(e) $|a+b| \leq |a| + |b|$

Items (a)–(d) are easy to see valid by the definition. Case (e) is commonly referred as **triangle inequality**. We have that $a \leq |a|$ and $-a \leq |a|$ for any a. This gives

$$a + b \leq |a| + |b|$$

and

$$-(a+b) = -a + (-b) \leq |a| + |b|$$

Because $|a+b|$ is either $a+b$ or $-(a+b)$, the claim follows.

Example 6 (Polynomial functions)**.** A polynomial function is a function involving only non-negative integer powers of x. We can give a general definition of a polynomial, and define its degree.

A **polynomial** of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where the a's are real numbers called the **coefficients** of the polynomial.

A polynomial can be also expressed using **summation notation**:

$$\sum_{k=0}^{n} a_k x^k.$$

Although these general formulas can look quite complicated, particular examples are much simpler. For example,

$$f(x) = 4x^3 - 3x^2 + 2$$

is a polynomial of degree 3, as 3 is the highest power of x in the formula. This is called a cubic polynomial, or just a cubic. And

$$f(x) = x^7 - 4x^5 + 1$$

is a polynomial of degree 7, as 7 is the highest power of x.

The **function composition** is an operation that takes two functions f and g and produces a function h such that $h(x) = g(f(x))$. The function g is applied to the result of applying the function f to x. The functions $f\colon X \to Y$ and $g\colon Y \to Z$ are composed to yield a function that maps $x \in X$ to $g(f(x)) \in Z$.

The resulting composite function is denoted $g \circ f\colon X \to Z$, defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$. The notation $g \circ f$ is read as "g circle f"

Composing functions is a chaining process in which the "output" of function f is given as "input" of function g. The composition of functions is a special case of the composition of relations.

Example 7. Let $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, and $C = \{x, y, z\}$. Suppose that the functions f and g are defined as

$$f(a) = 1,\ f(b) = 2,\ f(c) = 2; \qquad g(1) = x,\ g(2) = z,\ g(3) = y,\ g(4) = y$$

Then the composition $g \circ f$ is a function $A \to C$ such that

$$\begin{aligned}(g \circ f)(a) &= g(f(a)) = g(1) = x,\\ (g \circ f)(b) &= g(f(b)) = g(2) = z,\\ (g \circ f)(c) &= g(f(c)) = g(2) = z.\end{aligned}$$

Example 8. Let X be a set. The **identity function** $\mathrm{id}_X\colon X \to X$ is a mapping which maps each element x of X to itself, that is,

$$\mathrm{id}_X(a) = a$$

for all $a \in X$.

For any function $f\colon X \to X$, we can define $f^n\colon X \to X$ as the n-th **power** of f, where $n \in \mathbb{N}$, by setting

$$f^0 = \mathrm{id}_X \quad \text{and} \quad f^{n+1} = f \circ f^n.$$

Let f be a function from a set A to B.

- The function f is **injective**, or **one-to-one**, if distinct elements of A have different images. An injective function is also called an **injection**. Formally, this condition is presented as:

  $$(\forall x, y \in A)\, x \neq y \implies f(x) \neq f(x').$$

  This is equivalent to

  $$(\forall x, x' \in A)\, f(x) = f(x') \implies x = x'.$$

- The function f is **surjective**, or **onto**, if each element of A is mapped to by at least one element of B. A surjective function is a **surjection**. Formally the condition is presented as:

$$(\forall y \in B)(\exists x \in A) \text{ such that } y = f(x).$$

- The function f is **bijective** if it is both injective and surjective. A bijective function is also called a **bijection**. A bijection is also called a **one-to-one correspondence**.

Example 9. All maps here are from $\mathbb{Z}$ to $\mathbb{Z}$.

(1) The map $f(n) = n^2$ is not an injection and not a surjection.

(2) The map $g(n) = 2n$ is an injection, but not a surjection.

(3) The map $h(n) = n + 2$ is a bijection.

If $f\colon A \to B$ and $g\colon B \to A$ are functions, we say that g is an **inverse** to f (and f is an inverse to g) if and only if $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. The inverse of f (if it exists) is denoted by f^{-1}. It is clear that $f = (f^{-1})^{-1}$. In other words, if g is the inverse function of f, then f is the inverse function of g.

Theorem 10. *A function $f\colon A \to B$ has an inverse if and only if it is bijective.*

Proof. Suppose g is an inverse to f. If $f(a) = f(b)$, then $(g \circ f)(a) = (g \circ f)(b)$. Because $g \circ f = \mathrm{id}_A$, we have $a = b$ and f is injective. If $g \in B$, then because $f \circ g = \mathrm{id}_B$, then we can set $a = g(b)$. Now $f(a) = g(f(b)) = (g \circ f)(b) = b$ and f is surjective. We have now prove that f is bijective.

Conversely, suppose f is bijective. This means that for each $b \in B$, there exists exactly one $a \in A$ that satisfies the condition $f(a) = b$. We can define the function $g\colon B \to A$ by setting $g(b) = a$, where a is such that $f(a) = b$. Now

$$(f \circ g)(b) = f(g(b)) = f(a) = b \qquad \text{and} \qquad (g \circ f)(a) = g(f(a)) = g(b) = a.$$

Thus, g is the inverse of g. □

The inverse function essentially undoes the effects of the original function.

(i) Addition is defined in the whole $\mathbb{R}$ and it is “undone” by subtraction: when you add 5 to x, you get $x + 5$; to reverse this operation you need to subtract 5 from $x + 5$. More precisely, if $f(x) = x + 5$, then $f^{-1}(x) = x - 5$. This is because $f^{-1}(f(x)) = f^{-1}(x+5) = (x+5)-5 = x$ and $f(f^{-1}(x)) = f(x-5) = (x-5)+5 = x$.

(ii) Multiplication is also defined in whole $\mathbb{R}$ and it is "undone" by division: if you multiply x by 4, you get $4x$. You then must divide $4x$ by 4 to return to the original expression x. Indeed, if $f(x) = 4x$, then $f^{-1}(x) = x/4$. This is because $f^{-1}(f(x)) = f^{-1}(4x) = (4x)/4 = x$ and $f(f^{-1}(x)) = f(x/4) = 4 \cdot (x/4) = x$.

(iii) Let us now restrict ourselves to non-negative real numbers denoted by $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$. By definition, the *root function is the inverse of power function*, because for any $a, p, x \in \mathbb{R}^+$,

$$x = \sqrt[p]{a} \iff x^p = a.$$

Now $(\sqrt[p]{a})^p = (a^{\frac{1}{p}})^p = a^{\frac{1}{p} \cdot p} = a^1 = a$ and $\sqrt[p]{a^p} = (a^p)^{1/p} = a^{p \cdot \frac{1}{p}} = a^1 = a$.

(iv) **Logarithm** is the *inverse* of exponentiation. That means the logarithm of a given number x is the exponent to which another fixed number, the base b, must be raised, to produce that number x.

For instance, since $1000 = 10 \times 10 \times 10 = 10^3$, the logarithm base 10 of 1000 is 3, or $\log_{10}(1000) = 3$. The logarithm of x to base b is denoted as $\log_b(x)$, or without parentheses, $\log_b x$.

More precisely, the relation between exponentiation and logarithm is:

$$\log_b(x) = y \iff b^y = x, \text{ where } x > 0 \text{ and } b > 0 \text{ and } b \neq 1$$

For example, $\log_2 64 = 6$, because $2^6 = 64$. Note also that

$$\log_b b^y = y.$$

Proposition 11. (a) $\log_b(xy) = \log_b x + \log_b y$ *(product)*

(b) $\log_b \frac{x}{y} = \log_b x - \log_b y$ *(quotient)*

(c) $\log_b \left(x^k\right) = k \log_b x$ *(power)*

(d) $\log_a x = \frac{\log_b x}{\log_b a}$ *(changing base)*

Proof. (a) Let $m = \log_b x$ and $n = \log_b y$. We have $x = b^m$ and $y = b^n$. Therefore,

$$xy = b^m \cdot b^n = b^{m+n}.$$

We have that

$$\log_b(xy) = \log_b(b^{m+n}) = m + n = \log_b x + \log_b y$$

(b) Similarly as in (b), let $m = \log_b x$ and $n = \log_b y$. We have $x = b^m$ and $y = b^n$. Thus,

$$\frac{x}{y} = \frac{b^m}{b^n} = b^{m-n}.$$

We have

$$\log_b \frac{x}{y} = \log_b b^{m-n} = m - n = \log_b x - \log_b y$$

(c) Suppose Suppose $m = \log_b x$. Then, $x = b^m$. Raise both sides to the power k, we have $x^k = (b^m)^k = b^{km}$. Take logarithms both sides:

$$\log_b x^k = \log_b b^{km} = km = k \log_b x.$$

(d) Let $k = \log_a x$. Therefore, $x = a^k$. Take logarithms both sides:

$$\log_b x = \log_b a^k = k \log_b a$$

We have

$$\log_a x = k = \frac{\log_b x}{\log_b a}.$$

□

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The **cardinality of a set** is a measure of the "number of elements" of the set. For example, the set $A = \{2, 4, 6\}$ contains 3 elements, and therefore A has a cardinality of 3. The cardinality of a set A is usually denoted $|A|$; this is the same notation as absolute value.

For example, if $X = \{1, 2, 3, \ldots, 9\}$, then $|X| = 9$. However, it is clear that the cardinality of all sets cannot be given by a natural number. For example, the set $\mathbb{N}$ is like this. We will return to this later in this section.

Definition 12. A set A is **finite** if $|X| = n$ for some $n \in \mathbb{N}$. A set is **infinite** if it is not finite. If A is an infinite set, then we denote $|A| = \infty$.

This means that each set is either finite or infinite, but not both at the same time.

Example 13. (a) We show that the set of natural numbers $\mathbb{N}$ is infinite. We assume for contradiction that $\mathbb{N}$ is finite. Then there is a number $n \in \mathbb{N}$ such that $|\mathbb{N}| = n$ and $\mathbb{N} = \{x_1, x_2, \ldots, x_n\}$.

The number $x = x_1 + x_2 + \cdots + x_n + 1$ is a natural number. In addition, n is bigger than any of the numbers x_i. This means that the set $\{x_1, x_2, \ldots, x_n\}$ cannot contain all natural numbers, a contradiction! Therefore, the set $\mathbb{N}$ is not finite. So, it is infinite.

(b) We show that the set of prime numbers is infinite. We start noting that each natural number $n \geq 2$ can be presented as a *product* of primes. We will prove this result later in the course while studying number theory, but now we just need to believe that it is true. For instance,

$$825 = 3 \times 5 \times 5 \times 11.$$

Therefore, n is prime if and only if the product consists of the number n only.

Let us assume that the set of primes is finite. This means that there is $n \in \mathbb{N}$ such that the set $\{p_1, p_2, \ldots, p_n\}$ contains all primes.

Let us consider the natural number

$$x = p_1 p_2 \cdots p_n + 1$$

Let us now divide x by a prime number by p_i.

$$\frac{x}{p_i} = p_1 p_2 \cdots p_{i-1} \cdot p_{i+1} \cdots p_n + \frac{1}{p_i}.$$

The result is not an integer.

Because x is not divisible by any of the known prime numbers p_i, x itself must be a new prime number. But $x \notin \{p_1, p_2, \ldots, p_n\}$, a contradiction. Therefore, the set of primes is infinite.

Two sets A and B have the **same cardinality** if there exists a bijection from A to B.

Example 14. The size of the even numbers has the same cardinality as $\mathbb{N}$. This is because the mapping $f(n) = 2n$ is a bijection from the set $\{0, 1, 2, 3, \ldots\}$ to the set $\{0, 2, 4, 6, \ldots\}$.

According to the previous example, there are sets that have the same cardinality as their own proper subset! Clearly finite sets cannot have such a feature.

Definition 15. A **countable set** is a set with the same cardinality as some subset of the set of natural numbers. A countable set is either a **finite** set or a **countably infinite** set. If A is countable infinite, then we write $|A| = \aleph_0$. An infinite set is **uncountable** if it is not countable.

The letter $\aleph$ is pronounced *aleph.* If $|A| = \aleph_0$, we may think that A's elements are ordered in infinite sequence

$$a_0, a_1, a_2, \ldots$$

such that each element of A appears **sooner or later**. If A is uncountable, then the elements of A cannot be enumerated in this way.

Lemma 16. *Let A and B countably infinite sets. Then*

(a) $A \cup B$ *is countably infinite.*

(b) $A \times B$ *is countably infinite.*

Proof. (a) Because A and B are countable, we can assume that we can enumerate their elements as

$$a_0, a_1, a_2, \ldots \quad \text{and} \quad b_0, b_1, b_2, \ldots$$

We get an enumeration for $A \cup B$

$$a_0, b_0, a_1, b_1, a_2, b_2 \ldots$$

by leaving out possible duplicates; they may occur if $A \cap B \neq \emptyset$.

(b) The following diagram gives an enumeration for $A \times B$:

$$(a_0, b_0), (a_1, b_0), (a_0, b_1), (a_0, b_2), (a_1, b_1), (a_2, b_0), \ldots$$

$A \setminus B$	b_0		b_1		b_2	$\cdots$
a_0	(a_0, b_0)		(a_0, b_1)	$\rightarrow$	(a_0, b_2)	$\cdots$
	$\downarrow$	$\nearrow$		$\swarrow$		
a_1	(a_1, b_0)		(a_1, b_1)		(a_1, b_2)	$\cdots$
		$\swarrow$		$\nearrow$		
a_2	(a_2, b_0)		(a_2, b_1)		(a_2, b_2)	$\cdots$
	$\downarrow$	$\nearrow$		$\swarrow$		
a_3	(a_3, b_0)		(a_3, b_1)		(a_3, b_2)	$\cdots$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\ddots$

In the kth diagonal, we enumerate such elements (a_i, b_j) that $k = i + j - 1$. This means that we first enumerate pairs (a_i, b_j) such that $i + j = 0$, then those that satisfy $i + j = 1$, then the ones with $i + j = 2$, and so on. □

Example 17. Using the Proposition 16, we may show that

1. the set $\mathbb{Z}$ of all integers is countable, and
2. the set of all rational numbers $\mathbb{Q}$ is countable.

We have shown that the sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are similar in the sense that they all have the cardinality $\aleph_0$. Next, we show that there are uncountable sets.

Example 18. Let us determine the cardinality of the set $\wp(\mathbb{N})$ of all subsets of $\mathbb{N}$. We prove by contradiction that $\wp(\mathbb{N})$ is uncountable.

Let us assume that $\wp(\mathbb{N})$ is countable. This means that we can enumerate all subset of $\mathbb{N}$ as

$$X_0, X_1, X_2, X_3, \ldots$$

Next we show that all subsets cannot be in this list. This is done by colouring each natural number $i \in \mathbb{N}$ by the rules:

- the number i is **red**, if $i \in X_i$;
- the number i is **black**, if $i \notin X_i$.

Let us denote by $\mathbb{B}$ the set of all black numbers. It is clear that $\mathbb{B} \in \wp(\mathbb{N})$. Because $\wp(\mathbb{N})$ is countable, there exists a number $k \in \mathbb{N}$ such that $\mathbb{B} = X_k$.

Since k is a natural number, it should have a color. But both possible colors lead to a contradiction!

- If k is black, then $k \in \mathbb{B} = X_k$ and the colour of k is red, a contradiction!
- If k is red, then $k \in X_k = \mathbb{B}$, which means that the colour of k is black, a contradiction!

We have now show that the assumption '$\wp(\mathbb{N})$ is countable' leads always to a contradiction. Therefore, $\wp(\mathbb{N})$ is uncountable.

Another way to show that $\wp(\mathbb{N})$ is uncountable is the following. Each subset $X \subseteq \mathbb{N}$ can be identified by an infinite vector v such that ith element of v is 1 if i belongs to X, and 0 otherwise. Suppose that $\wp(\mathbb{N})$ has an enumeration:

$$\begin{aligned}
S1 &= (\mathbf{0}, 0, 0, 0, 0, 0, 0, \ldots) \\
S2 &= (1, \mathbf{1}, 1, 1, 1, 1, 1, \ldots) \\
S3 &= (0, 1, \mathbf{0}, 1, 0, 1, 0, \ldots) \\
S4 &= (1, 0, 1, \mathbf{0}, 1, 0, 1, \ldots) \\
S5 &= (1, 1, 0, 1, \mathbf{0}, 1, 1, \ldots) \\
S6 &= (0, 0, 1, 1, 0, \mathbf{1}, 1, \ldots) \\
S7 &= (1, 0, 0, 0, 1, 0, \mathbf{0}, \ldots) \\
&\vdots
\end{aligned}$$

Next, we *generate an infinite vector* S by choosing the 1st digit as complementary to the 1st digit of $S1$ (swapping 0s for 1s and vice versa), the 2nd digit as complementary to the 2nd digit of $S2$, the 3rd digit as complementary to the 3rd digit of $S3$, and generally for every i, the ith digit as complementary to the ith digit of Si. For the example above, we get

$$S = (1, 0, 1, 1, 1, 0, 1, \ldots).$$

By construction, S differs from each Si, since their ith digits differ (highlighted in the example). Hence, S cannot occur in the enumeration.

Next we will show that a *rational number* can be represented in decimal form as a terminating or nonterminating recurring decimal. For example

$$5/2 = 2.5, \quad 2/8 = 0.25, \quad 7 = 7.0,$$

are rational numbers which are terminating decimals. On the other hand,

$$5/9 = 0.555555555\cdots, \quad 4/3 = 1.33333\cdots, \quad 9/11 = 0.81818$$

are rational numbers which are nonterminating, recurring decimals.

Lemma 19. *A real number is rational if and only if it has terminating or recurring decimal.*

Proof. Let us first consider how we can find the "rational form" of a decimal number n.

For instance, $n = 1.2367$ has a *terminating decimal* of length 4. We simply multiply n by 10^4 and divide the product by 10^4, so n does not change and get

$$n = \frac{12367}{10000}.$$

Consider the *repeating decimal* $n = 2.135135135\cdots$. The repeating part (135) is 3 digits long, so we multiply n by 10^3 to get

$$1000n = 2135.135\,135\,135\cdots$$

If we subtract n from this, we get

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1000 n = 2135.135 135 135 ...
     n =    2.135 135 135 ...
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 999 n = 2133
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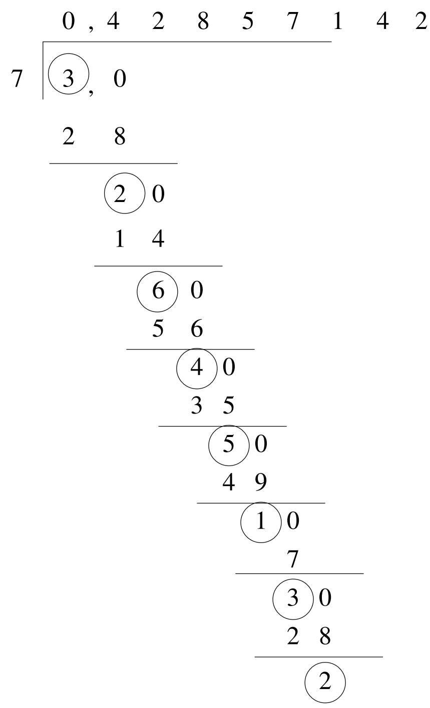
and so $n = 2133/999$. Because both the numerator and denominator can be divided by 27, we can write n in a simplified form

$$n = \frac{79}{37}.$$

We have now described how to find the decimal corresponding to a terminating or recurring decimal number. On the other hand, let us consider the rational $x = \frac{3}{7}$.

To find out the decimal, we need to do the division:



Remainders are marked with circles. Because we are dividing by 7, the remainder can be only one of the numbers 0, 1, 2, 3, 4, 5, 6 (do not worry, we consider basics of the number theory later in this course). When the same number (different from 0) appears as remainder for the second time, the repetition necessarily begins.

Note that our proof is not *valid* in that sense that we only considered special cases. But it should be noted that an analogous proof could be presented in a general form. □

We have proved that

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769480731766797379$$

is not rational, and its decimal is not recurring. This implies that also

$$\frac{1}{\sqrt{2}} = 0,7071067811865475244008443621048490392848359376884740365883398689 95\cdots$$

is not rational and its decimal is not recurring.

Example 20. We prove that the set of real numbers

$$(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$$

is not countable. This implies that $\mathbb{R}$ is not countable.

Let us first note that real number $0 < x < 1$ has a unique decimal representation. If x is a rational number, as we have noted, the decimal can be terminating or recurring decimal number. Recurring terminals are naturally of infinite length. A terminating number, like 0.5 can be made infinite just by adding infinite number of zeros in the end, that is,

$$0.5000000000000\cdots$$

Note that terminating numbers could be represented also in a different form. For instance, 0.5 could be represented also in the form:

$$0.4999999999999999999999999\cdots$$

(This is because there is "nothing" between 0.5 and $0.4999999999999\cdots$). Therefore, the representations having a tail-end that consists entirely of the digit 9 are excluded.

Suppose that $(0,1)$ is countable. Clearly $(0,1)$ is not a finite set, so we are assuming that $(0,1)$ is countably infinite. Then there exists a bijection from $\mathbb{N}$ to $(0,1)$. In other words, we can create an infinite list which contains every real number. Write each number in the list in decimal notation. Such a list might look something like:

$$\begin{array}{ll} 1 & 0.0234242420905903943493 4\cdots \\ 2 & 0.3243429342942949243924 2\cdots \\ 3 & 0.5000000000000000000000 0\cdots \\ 4 & 0.2034230492094029402949 0\cdots \end{array}$$

Under this assumption, the real numbers between 0 and 1 can be listed in some order, say, a_1, a_2, a_3,... Let the decimal representation of these real numbers be:

$$\begin{array}{l} 0, a_{11}a_{12}a_{13}a_{14}a_{15}\cdots \\ 0, a_{21}a_{22}a_{23}a_{24}a_{25}\cdots \\ 0, a_{31}a_{32}a_{33}a_{34}a_{35}\cdots \\ 0, a_{41}a_{42}a_{43}a_{44}a_{45}\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ 0, a_{k1}a_{k2}a_{k3}a_{k4}a_{k5}\cdots \\ \cdots\cdots\cdots\cdots\cdots \end{array}$$

Let us consider the number (diagonal):

$$0, a_{11}a_{22}a_{33}a_{44}a_{55}\cdots$$

Then, form a new real number with decimal expansion

$$b = 0, b_1b_2b_3b_4b_5\cdots$$

where the decimal digits are determined by the following rule:

$$b_i = \begin{cases} 4 & \text{if } a_{ii} \neq 4 \\ 5 & \text{if } a_{ii} = 4 \end{cases}$$

Therefore, the real number b is not equal to any of $a_1, a_2, a_3, \ldots$ because the decimal expansion of b differs from the decimal expansion of a_i in the ith place to the right of the decimal point, for each i. Because there is a real number b between 0 and 1 that is not in the list, the assumption that all the real numbers between 0 and 1 could be listed must be false. Therefore, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable. Any set with an uncountable subset is uncountable. Hence, the set of real numbers is uncountable

Example 21. Consider the function

$$f(x) = \begin{cases} \dfrac{1}{x} - 2 & \text{if } 0 < x \leq \dfrac{1}{2} \\ \dfrac{1}{x-1} + 2 & \text{if } \dfrac{1}{2} < x < 1 \end{cases}$$

It can be shown that $f\colon (0,1) \to \mathbb{R}$ is a bijection. This means that $(0,1)$ contains as many numbers as the whole $\mathbb{R}$!

Example 22. Is the set of complex numbers $\mathbb{C}$ bigger than $\mathbb{R}$?

Example 23. Even we have shows that the number of rationals is equal to the number of integers, between two real numbers there are always one rational!

We may assume that x and y are two distinct positive real numbers. For instance, if x is negative and y is positive, then the claim is obvious. Also negative numbers behave like positive numbers.

Let x and y be the following two numbers:

```
1.414213562373095048801688724209698078569671875376948073171...
1.414213562373095048801688724209698078569671875376948073183...
```

We see that the endings of the two numbers are different x ends with $\cdots 73171 \cdots$ and y ends with $\cdots 73183 \cdots$ We can now always select a terminating ending between these two, for instance $\cdots 73179$. So, the terminating decimal number (rational)

```
1.414213562373095048801688724209698078569671875376948073179
```

is a rational between x and y.